INTEGRATING TRANSPORTATION AND INVENTORY DECISIONS IN A MULTI-WAREHOUSE MULTI-RETAILER SYSTEM WITH STOCHASTIC DEMAND

By

PAVEENA CHAOVALITWONGSE

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To my parents, Vinij and Boonya Chaovalitwongse.

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Paveena Chaovalitwongse

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Chairman: Dr. Boghos D. Sivazlian
Cochair: Dr. Panos M. Pardalos
Major Department: Industrial and Systems Engineering

Two major cost components in distribution system are inventory and transportation cost. For convenience, the inventory and transportation decisions are determined as two disjointed decisions which result in a high system total cost. Unifying these two decisions can lead to reduction in the system total cost.

This dissertation considers the problem of multiple capacitated warehouses supplying multiple retailers with stochastic demand. Its purpose is to develop analytical models and solution approaches for optimizing a commodity flow in a distribution system under uncertain demand based on the sum of inventory and transportation costs. The research aims to establish improved decision-making methods that integrate inventory replenishing policy and transportation strategy.

The developed model formulation extends the simple single-period stochastic inventory problem (newsboy or newsvendor problem) to incorporate the transportation cost into the objective function. In addition, the capacity constraints are added to impose over-supplying at warehouses. Two forms of the transportation cost functions are considered: linear and fixed charge cost functions.

The Lagrange multiplier method is employed to solve the linear transportation cost model. This method works well only with small problems. In the fixed charge cost model, the dynamic slope scaling procedure (DSSP) is employed to develop the scenario-based heuristic. The DSSP is an effective and efficient approach to estimate an optimal solution to a fixed charge network flow problem. Based on a set of test problems, computational results show that the DSSP scenario-based heuristic obtains the solution in reasonable computational time. This obtained solution to the approximated problem (scenario-based model) is used to evaluate the true objective value in the original non-linear problem, which gives the upper bound to the original problem. Since the optimal solution to the original problem is unknown, the conclusion on the upper bound quality cannot be established. Alternatively, the gap between upper bound and lower bound can be used as a criterion. The best lower bound is obtained by the development of Lagrangian relaxation model. Computational experiments show that the performance of the DSSP heuristic is excellent, based on a set of test problems.

Finally, we improve the DSSP heuristic by employing the Lagrangian relaxation method to solve a linear transportation cost and eliminate scenario generating work. Based on a set of test problems, the Lagrangian relaxation-based heuristic provides a better solution in less computational time than the scenario-based heuristic.

CHAPTER 1
INTRODUCTION

1.1 Background and Motivation

Supply chain management (SCM) has become one of the most studied research areas in operations research. A supply chain involves a series of activities: purchasing, manufacturing, stocking, and distributing. This activity series represents a flow of commodities from procuring raw materials to delivering to end customers. Along the flow, one or more facilities of the following (vary from company to company) are involved: factories, distribution centers, warehouses, and retailers. Thus, SCM is the management of this flow which is equivalent to logistics management.

Embedding in distributing activities, inventories and transportation are two key elements in the logistics. Since the future is not certain, inventories are served as precautionary assets. They protect companies against commodity shortage which gradually destroys customer satisfaction, company's reputation, and company's market position. As for transportation, customers are not typically located in the same places as manufacturers. Thus, transportation is a crucial element in the supply chain to move commodities to the hand of customers which is the closing point of the commodity flow.

The combined inventory and transportation cost is equal to one-half to two-thirds of total logistics costs [8]. Robert V. Delaney reported in "State of Logistics Report" that in 1999 American business logistics costs were $921 billion (9.9 percent of nominal Gross Domestic Product). This figure had increased about 7 percent from 1997. Since

the inventory-transportation cost plays an important role in the logistics total cost, there is an increasing need for an efficient and effective distribution strategy in which the company's logistics total cost can be reduced. Determining such a strategy requires a well-developed analytical model and analysis techniques that coordinate inventory management and transport strategy.

The purpose of this dissertation is to develop analytical models and solution approaches for determining an efficient distribution strategy in a system of multiple warehouses supplying multiple retailers. The developed model aims to optimize a commodity flow so that the system total cost is minimized. The scope encompasses transportation cost of shipping commodities from warehouses to retailers via different transport channels (connection between a warehouse and a retailer), and inventory decisions (holding and shortage) at retailers. In this introductory chapter, a brief overview of a single period stochastic inventory problem and a transportation problem is presented in Section 1.2 and 1.3. The organization of the remainder of this dissertation is framed in Section 1.4.

1.2 The Single Period Stochastic Inventory Problem

As its name suggested, this problem deals with the one-period, periodic review inventory problem with a probabilistic demand. A classical example of this type is the newsboy or newsvender problem. It has the simplest model structure which considers only a single commodity, no fixed ordering cost, zero initial inventory, and no constraint. In this problem, a newsboy must order newspapers to sell at the beginning of a day. If the newsboy carries too many newspaper, then many worthless newspapers are left at

the end of the day. On the other hand, a newsboy who orders too few newspapers will lose opportunity to make additional profit. The question is how many newspaper should be ordered in order to maximize the newsboy's profit.

Applications of this static one-shot type decision often appear in the manufacturing of highly seasonal products such as toys, high-fashion clothes, and sports event souvenirs. Similar to the newsboy problem, the products may become obsolete at the end of the season if the production is exceeding the demand. On the other hand, a loss of profit may be induced by shortage of products resulting from a limited production. The management must wonder what would be the optimal production level so as to minimize their expected total cost or maximize their expected profit. An important key of this decision-making process is the uncertainty or randomness of demand. If the demand is known, then this decision would become easier to make.

The newsboy problem and its solution are presented in many introductory textbooks in operations research [54, 59], operations management, or inventory management [52, 47]. The closed-form formulations for computing the expected total cost are presented in Lau [36]. These formulations are derived with various demand distributions. The inventory replenishing policy for the newsboy problem is known as an order-up-to policy resulting from the tradeoff between holding (overstocking) and shortage (understocking) cost. Usually, the optimal order-up-to level is denoted by S^*. This problem is more complicated when a fixed cost is incurred for replenishment but not otherwise. This fixed cost may include an order processing cost, an administrative cost, and so on. Taking into account this fixed cost, the replenishing policy is modified to (s,S) policy. Under this policy, when the initial inventory level at the beginning of the period is less

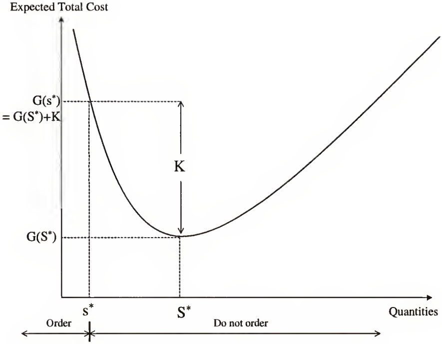


Figure 1.1: Expected cost function for newsboy model with fixed ordering cost

than s, the order is placed to bring its level to S. The (s,S) policy incorporates two decision variables: s and S. The variable s specifies when to make an order, while both variables s and S specify how many units to order.

Let $G(\bullet)$ and K denote the expected cost function and fixed ordering cost respectively. The optimal s^* is derived from the equation below.

$$G(s^*) = K + G(S^*)$$

where S^* can be derived by the same method in the order-up-to policy. Figure 1.1 shows that it is better not to order when the inventory level falls between (s^*, S^*). Otherwise, the fixed ordering cost would increase the total expected cost. As K decreases, s^*

becomes larger. At the point where K is zero, s^* is equal to S^*. It is noted that the order-up-to policy is a special case of (s,S) policy in which s is equal to S.

Many studies in the literature have been done to extend the simple newsboy problem. The investigation of multi-commodity or multi-product type system appears in several references [54, 45, 37, 38]. Some studies focus on incorporating constraints (resource restrictions) [52, 45, 37, 38]. Many contributions are devoted to the demand 'distribution free' approach. This approach uses the information of estimated mean and variance without the knowledge of demand distribution [18, 26, 44, 43]. Its goal attempts to decrease the value of missing information (demand distribution).

1.3 The Transportation Problem

The transportation problem (TP) is a special case of the minimum cost network flow problem (MCNFP). The MCNFP determines the least cost of distributing the commodities through a network such that demands at certain nodes are satisfied by available supplies of other nodes [2]. Different from the basic MCNFP, the TP has a bipartite network structure. That means the node set is partitioned into two subsets: demand node subset and supply node subset. A supply node is connected to a demand node by an arc. The flow on each arc has only one direction from the supply node to the demand node. Usually the demands are known and the supplies are finite at the supply node. One of applications of the TP is the distribution of products from warehouses to retailers. Suppose that a system consists of m warehouses supplying n retailers. Let i index the warehouses in the set $I = \{1, \ldots, m\}$ and j index the retailers in the set $J = \{1, \ldots, n\}$. A flow on each arc incurs a unit variable cost, c_{ij}. Let x_{ij} be the amount

shipped from warehouse i to retailer j. The available capacity at the warehouse i is s_i, while the amount required at the retailer j is d_j. Then the mathematical formulation of the TP can be given as follows:

$$\begin{aligned}
\min \quad & \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}x_{ij} & & & \\
s.t. \quad & \sum_{j=1}^{n} x_{ij} \;\leq\; s_i & \forall i & \quad \text{(Supply constraints)} \\
& \sum_{i=1}^{m} x_{ij} \;\geq\; s_j & \forall j & \quad \text{(Demand constraints)} \\
& x_{ij} \;\geq\; o & \forall ij & \quad \text{(Nonnegativity constraints)}
\end{aligned}$$

The objective is to minimize total cost of distributing commodities from supply nodes to demand nodes subject to all constraints. In supply constraints, the total amount shipped from each warehouse cannot exceed its capacity. In demand constraints, the total amount received at each retailer must satisfy its demand. The last constraint ensures that all flows cannot be negative. This problem can be solved efficiently (see [2] for several algorithms for transportation problems).

Another form of the TP is a fixed charge transportation problem (FCTP). Now the cost flow in each arc has two components: fixed and variable. The fixed component does not depend on the amount shipped, but rather on whether or not there is a shipment flow on that arc. In the logistics context, the fixed component may include the minimum freight charge for delivery of the commodities or an order processing charge. The FCTP belongs to the class of the minimum concave cost network flow problem (MCCNFP) which is an $\mathcal{NP}$-hard problem. This means there is no known algorithm which can solve this class of problem in polynomial time. Guisewite and Pardalos [30] survey general characteristics of the MCCNFP and its solution methods.

Transformed into a zero-one mixed integer programming problem, the FCTP can be solved using branch and bound algorithms to obtain an exact optimal solution. Many studies have been done to enhance the existing branch and bound based algorithms. One recent work by Bell et al. [11], develops a capacity improvement technique to improve their branch and bound algorithm.

1.4 Organization of the Dissertation

The material presented in this dissertation is organized into six chapters.

- Chapter 2 deals with the problem of deriving the optimal inventory replenishing policy. It begins with a review of various stochastic inventory problems of both single-echelon and multi-echelon systems. Then, it develops a single-period two-echelon stochastic model with a fixed ordering cost for a single-warehouse single-retailer system. The comprehensive analytical result is presented.
- Chapter 3 introduces transportation cost in a capacitated multi-warehouse multi-retailer single-period stochastic inventory problem. A review of the literature in minimizing inventory-transportation cost model is presented at the beginning. A nonlinear programming model is developed to minimize the inventory and transportation costs simultaneously. This model facilitates in determining an integrated inventory replenishing policy and a transporting plan. Developed first, the linear transportation cost model applies the Lagrange multiplier method for analyzing the optimal integrated policy.
- Chapter 4 substitutes the fixed charge transportation cost in the optimization model presented in Chapter 3. The fixed charge model optimal value is estimated

by the DSSP heuristic through a scenario-based approximated problem. The computational results for a wide variety of problem sizes are reported.

- Chapter 5 measures the quality of the upper bound obtained from Chapter 4 by comparing it with a lower bound. The best lower bound can be obtained from the development of a Lagrangian relaxation model. This relaxation model employs the subgradient optimization algorithm to compute the lower bound. The extensive comparison results are presented.
- Chapter 6 develops a new Lagrangian based DSSP heuristic. This approach employs the Lagrangian relaxation method to solve the linear transportation cost problem. As a result, it eliminates an approximation of demand distribution through the scenario generating process.
- Chapter 7 describes directions for future research. It surfaces a broad variety of research challenges to be treated in much greater depth.

CHAPTER 2
THE SINGLE-PERIOD TWO-ECHELON STOCHASTIC INVENTORY PROBLEM

2.1 Introduction

Most inventory systems in a supply chain are of a multi-echelon type, the management of which is complex and challenging. With this type of system, dependency among echelons cannot be ignored. In practice individual stages often minimize cost based on local information to develop simple models and policies. This approach may not minimize the total cost of the whole system. Research has also shown that policies based on local information can lead to distorted and amplified demand requirements at higher echelons [39]. Thus the dependence among echelons should be considered as a coordinated system, which we would like to operate at minimum cost.

While many works have been devoted to the multi-echelon inventory control problem, systems with uncertain demands have received attentions only in recent years. Thus, this chapter embarks on the multi-echelon stochastic inventory problem with analyzing a simple two-echelon inventory problem with probabilistic demand. The simple newsboy problem is modified to incorporate an additional echelon inventory and fixed ordering cost is present.

The material of this chapter is organized as follows. Section 2.2 provides a review of literature on supply chain management. Section 2.3 describes the framework of the single-period two-echelon stochastic inventory problem. Section 2.4 introduces notation, presents the mathematical formula, and derives the optimal policy. Finally, Section 2.5

illustrates the derivation of the optimal policy under the negative exponential demand distribution.

2.2 Literature Review

Recently, many researchers have turned their attention to supply chain management (SCM) [56]. A multi-echelon inventory system is one of the most interesting problem in SCM. In fact, the multi-echelon inventory system has been studied since the 1960s. The most classical work is that of Clark and Scarf [21]. Their model focuses on a serial multi-installation inventory system, and an optimal policy is determined for serial systems using a recursive computation approach. While the demand is originated at the lowest installation, no setup cost exists in any installation except the highest installation. The term "echelon" stock is introduced. The echelon stock is referred to the stock at any given installation plus stock in transit to or on hand at downstream installations.

The probabilistic multistage serial inventory system is extended by Chen and Zhen [20] to include setup costs. For echelon-stock $(r,\ nQ)$ policies, they propose an efficient heuristic for determining near-optimal control parameters in multistage serial systems with compound Poisson demands. Their algorithm starts with determining both lower and upper bound on the cost function by over- and under-charging a penalty cost to each upstream stage for holding inadequate stock. Then these bounds are minimized to obtain a heuristic solution, that is near optimal. In addition, they also provide an algorithm that ensures the optimal solution. However, this algorithm requires more computational effort than is practical using existing computing power.

Love [40] applies dynamic programming recursion to develop an algorithm for finding an optimal nested schedule, for an n-period single-product inventory model. The model is N-facility serial inventory system with deterministic demand and separable concave production and storage costs. The demand occurs at the lowest echelon (labeled N). The nested schedule implies that if there is a production at facility j for a given period i, then all its successors must produce in period i. To guarantee the existence of a "nested" optimal schedule, the storage costs are not decreasing as the product flows down stream and the production costs are not increasing in time.

A relatively sophisticated model of a two-level inventory system of a single-item source supplying n parallel warehouses with probabilistic demand and no setup cost is investigated by Gross [29]. This model does not require the same demand distribution for all warehouses. Transshipping between warehouses is considered, because it might be more economical to transfer inventory from overloaded warehouses to understocked ones instead of ordering from the central source. The model neglects the lead-time between placing and receiving an order and that for initiating and receiving transshipment. Thus the total expected system costs are associated with variable ordering cost, transshipment cost, and holding and shortage costs. To minimize the expected total system costs, an iterative procedure is applied to a one-period model to provide a practical solution.

The two-level inventory system with a single source supplying n parallel depots is extended by Mahoney and Sivazlian [41] to incorporate setup costs and interdependence among the depots. Under a joint ordering policy or (σ, S) policy, the demand at each depot is assumed to have an Erlang distribution, and no transshipment between depots

is allowed. Given an order-up-to level, S_i's, and the "reorder" level, an expression for steady state stockout probability at each depot is derived.

Ernst and Pyke [22] examine a two-level distribution system composed of a warehouse and a retailer, which faces random demand. The model considers both inventory and transportation costs. Thus the expected total cost function is defined as holding and stockout costs at the warehouse and the retailer, and the transportation cost for shipping goods from warehouse to retailer. An algorithm is given that determines the optimal base stock policy at the retailer and the warehouse, the optimal in-house and contracted truck capacity, and the optimal review period length.

The study of a three-level distribution system appears in Chan and Simchi-Levi [19]. Their system consists of a single outside vendor, a fixed number of warehouses and many geographically dispersed retailers. The study offers an efficient algorithm for three level distribution systems so that long-run average cost is as small as possible. The long-run average cost is defined as the total inventory holding and transportation costs (from the outside vendor to the warehouses and from the warehouses to the retailers). The distribution strategy does not keep stock at the warehouses, a strategy often referred as "cross-docking." In other words, as soon as a fully loaded truck from the vendor arrives at each warehouse, the warehouse loads the goods onto other trucks, which deliver consolidated shipments of various items to retailers. Each retailer faces a deterministic demand rate, which is assumed identical over all retailers. No shortage or backlogging is allowed. Initially, each retailer is assigned to a single warehouse. This assignment is similar to a bin-packing problem where the capacity of the warehouse is analogous to the bin capacity and retailers are equivalent to items that require packing in the bins.

Retailers assigned to the same warehouse require the same delivery schedule. Retailers at each warehouse are then partitioned into clusters. Retailers in the same cluster receive shipments on the same truck. The partitioning problem is similar to finding an optimal traveling salesman tour.

According to Lee, Padmanabhan, and Whang [39], a demand variation phenomenon under stochastic environment in a serial supply chain is called "bullwhip" effect. In general, small variations or nearly steady consumer demand are amplified into high variation at higher echelon in the chain. By investigating the amplification of inter-echelon order variation in multi-echelon supply systems, Burns and Sivazlian [16] suggest two sources that create this amplification: (1) a legitimate and unavoidable inventory adjustment and (2) an unwarranted "false order" effect. A new decision rule is proposed to suppress false orders and avoid stockouts.

As reviewed in Sivazlian [53] and Sivazlian and Stanfel [54], literature relevant to our analysis offers optimal ordering strategies and some characteristics of a multi-commodity inventory systems with probabilistic or uncertain demands over a single period. Interdependence is assumed among the demands of commodities as well as the set-up costs. In the analysis, the marginal distributions of each commodity are only needed rather than the joint density distribution. The analysis is initiated with a two-commodity inventory system. The authors then extend their analysis to an m-commodity inventory system (m = 1, 2,...). In an m-commodity system, the number of possible ordering decisions equals 2^m, with an order being placed in $2^m - 1$ of these cases. Thus the objective is to determine the minimum expected total cost among those ordering decisions in a single period.

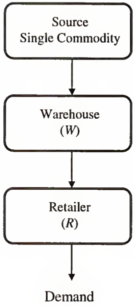


Figure 2.1: A single-commodity two-echelon system in series

2.3 Problem Description

Consider a single-period single-commodity inventory system consisting of three levels operating in series as shown in Figure 2.1. The system is composed of one retailer, one warehouse, and one source with infinite supplies. During a period, depletion of inventory at the retailer is induced by customer demand (D), while inventory depletion at the warehouse (D_W) is only induced by requests from the retailer. Lead-time at both warehouses is neglected. For simplicity, we assume that the demand at warehouse equals demand at retailer ($D_W = D$). Let $\varphi_D(\bullet)$ be a density function of customer demand and let $\Phi_D(\xi) = \int_0^\xi \varphi_D(u)\, du$ and $\bar{D} = \int_0^\infty u\varphi_D(u)\, du$.

We also assume that the customer demand must always be met. In other words, the unfulfilled demand is backordered, at some penalty cost per unit associated with it. One

important assumption to be made is to use echelon stock level instead of installation stock level. The echelon stock at the warehouse is defined as the stock at installation warehouse plus the stock at echelon retailer. The echelon stock at the retailer is the same as installation stock because it is at the lowest echelon of the system.

2.4 Model Formulation and Analytical Results

Now we define some notations.

Notation

c_R variable procurement cost at the retailer in \$/(unit)

c_W variable procurement cost at the warehouse in \$/(unit)

K_R fixed order cost at the retailer in \$

K_W fixed order cost at the warehouse in \$

h_R^* holding cost at installation retailer in \$/[(unit)(unit time)]

h_R holding cost at echelon retailer in \$/[(unit)(unit time)]

h_W^* holding cost at installation warehouse in \$/[(unit)(unit time)]

h_W holding cost at echelon warehouse in \$/[(unit)(unit time)]

p_R^* shortage cost at installation retailer in \$/[(unit)(unit time)]

p_R shortage cost at echelon retailer in \$/[(unit)(unit time)]

p_W^* shortage cost at installation warehouse in \$/[(unit)(unit time)]

p_W shortage cost at echelon warehouse in \$/[(unit)(unit time)]

x_R inventory level prior to making a decision at echelon retailer

(x_R can be negative)

x_W inventory level prior to making a decision at echelon warehouse

(x_W can be negative)

z_R quantity received at echelon retailer ($z_R \geq 0$)

z_W quantity received at echelon warehouse ($z_W \geq 0$)

y_R inventory level following a decision at echelon retailer

($y_R = x_R + z_R$)

y_W inventory level following a decision at echelon warehouse

($y_W = x_W + z_W$)

For this particular system, when a decision for replenishing stock is to be made, there are 4 possible courses of action or ordering decisions available. These are

1. order for the retailer only,
2. order for the warehouse only,
3. order for both warehouses, and
4. do not order anything.

2.4.1 Cost Structure of the Problem

We now develop a cost function for inventory that includes procurement cost and holding and shortage costs.

Procurement cost

For each warehouse, the two basic components of the procurement cost are a fixed cost and a variable cost. The fixed cost is incurred only if an order for replenishment is placed. The variable cost depends on the quantity received at each warehouse. Let

$P(x_R, x_W)$ denote the procurement cost function; then

$$P(x_R, x_W) = k(z_R, z_W) + c_R z_R + c_W z_W \tag{2.1}$$

where $k(z_R, z_W)$ represent the fixed portion of the procurement cost, and $c_R z_R$ and $c_W z_W$ denote the variable procurement costs at the retailer and the warehouse respectively. The fixed portion of the procurement cost may include the delivery cost such as a minimum freight charge, the driver's wages, the cost of processing an order, and so forth. The variable procurement cost is the cost per unit of commodity and delivery, excluding the fixed portion. We assume that the variable cost per unit at the warehouse is strictly less expensive than at the retailer. That is $c_R > c_W$. In general $k(z_R, z_W)$ can take many values depending on the aforementioned ordering decisions. Thus we can define it as follows:

$$k(z_R, z_W) = K_R \delta_R + K_W \delta_W \tag{2.2}$$

where $\delta_R = 1$ if $z_R > 0$, otherwise 0, and $\delta_W = 1$ if $z_W > 0$, otherwise 0.

Holding and shortage costs

At the end of a period, there may be an amount of stock left over or a shortage of inventory at either or both warehouses. The total expected holding and shortage cost incurred over the period at both warehouses is a function of the echelon stock level (y_R, y_W) following the decision process at the start of the period: let $L(y_R, y_W)$ denote the expected holding and shortage costs at the warehouse and the retailer. Thus the magnitude of $L(y_R, y_W)$ is directly effected by the initial echelon inventory level (x_R, x_W)

and the ordered amount (z_R, z_W), which varies based on the ordering decision. Since we use echelon stock as our decisions, the holding cost and shortage cost should be charged based on echelon costs. The echelon holding and shortage costs at the warehouse are defined as follows:

$$h_W = h_W^*$$

$$p_W = p_W^*.$$

Since a part of holding and shortage cost at the retailer have already been charged at the warehouse. Thus, the echelon holding and shortage costs at the retailer are defined as follows:

$$h_R = h_R^* - h_W^*$$

$$p_R = p_R^* - p_W^*.$$

Now, we can express $L(y_R, y_W)$ as follows.

$$\begin{aligned} L(y_R, y_W) &= h_R \int_0^{y_R} (y_R - \xi)\varphi_D(\xi)\, d\xi + p_R \int_{y_R}^{\infty} (\xi - y_R)\varphi_D(\xi)\, d\xi \\ &\quad + h_W \int_0^{y_W} (y_W - \xi)\varphi_D(\xi)\, d\xi + p_W \int_{y_W}^{\infty} (\xi - y_W)\varphi_D(\xi)\, d\xi \end{aligned} \tag{2.3}$$

The function $L(y_R, y_W)$ is separable in y_R and y_W. This means that we can express $L(y_R, y_W)$ as the sum of a function of y_R only and of a function of y_W only:

$$L(y_R, y_W) = L(y_R) + L(y_W) \tag{2.4}$$

where $L(y_R)$ and $L(y_W)$ can be expressed as follows:

$$L(y_R) = h_R \int_0^{y_R} (y_R - \xi)\varphi_D(\xi)\, d\xi + p_R \int_{y_R}^{\infty} (\xi - y_R)\varphi_D(\xi)\, d\xi \tag{2.5}$$

$$L(y_W) = h_W \int_0^{y_W} (y_W - \xi)\varphi_D(\xi)\, d\xi + p_W \int_{y_W}^{\infty} (\xi - y_W)\varphi_D(\xi)\, d\xi \tag{2.6}$$

It is interesting to note that both $L(y_R)$ and $L(y_W)$ are twice differentiable at all points y_R and $y_W > 0$ for which $\varphi_D(\bullet)$ is continuous. We note that $L''(y_R) = (h_R + p_R)\varphi_D(y_R)$ and $L''(y_W) = (h_W + p_W)\varphi_D(y_W)$. Thus, $L(y_R, y_W)$ is strictly convex in y_R and y_W.

2.4.2 Objective Function

Now we wish to determine an optimal replenishment policy so as to minimize the sum total of the procurement cost $P(x_R, x_W)$ and the expected holding and shortage $L(y_R, y_W)$ costs during a single period. Let $C(x_R, x_W)$ denote the minimum total expected cost for the period when following an optimal policy. Then $C(x_R, x_W)$ can be expressed as follows.

$$\begin{aligned} C(x_R, x_W) &= \min_{y_R \geq x_R, y_W \geq x_W} \{P(x_R, x_W) + L(y_R, y_W)\} \\ &= \min_{y_R \geq x_R, y_W \geq x_W} \{k(y_R - x_R, y_W - x_W) + c_R(y_R - x_R) \\ &\qquad + c_W(y_W - x_W) + L(y_R, y_W)\}. \end{aligned} \tag{2.7}$$

Define

$$G(y_R, y_W) = c_R y_R + c_W y_W + L(y_R, y_W). \tag{2.8}$$

Then

$$C(x_R, x_W) = \min_{y_R \geq x_R, y_W \geq x_W} \{k(y_R - x_R, y_W - x_W) - c_R x_R - c_W x_W + G(y_R, y_W)\}. \quad (2.9)$$

The function $G(y_R, y_W)$ plays an important role in determining optimal ordering decisions. As defined by relation (2.8), it is evident that if $L(y_R, y_W)$ is strictly convex, so will be the function $G(y_R, y_W)$. As defined, $G(y_R, y_W)$ will possess an absolute minimum occurring at a point (S_R, S_W), such that at $y_R = S_R$ and $y_W = S_W$ where $\partial G(y_R, y_W)/y_R = 0 = \partial G(y_R, y_W)/y_W$. The existence of (S_R, S_W) is guaranteed because of the strictly convexity of $G(y_R, y_W)$.

Now we want to find s_R and s_W. From (3.6) and (2.8), we can express $G(y_R, y_W)$ as follows:

$$G(y_R, y_W) = c_R y_R + c_W y_W + L(y_R) + L(y_W)$$

or

$$G(y_R, y_W) = G(y_R) + G(y_W). \quad (2.10)$$

Once we know (S_R, S_W), we can find s_R and s_W from the equations below.

$$\begin{aligned} G(s_R, y_W) &= K_R + G(S_R, y_W) \\ G(y_R, s_W) &= K_W + G(y_R, S_W) \end{aligned}$$

or

$$G(s_R) - G(S_R) = K_R \quad (2.11)$$

$$G(s_W) - G(S_W) = K_W. \quad (2.12)$$

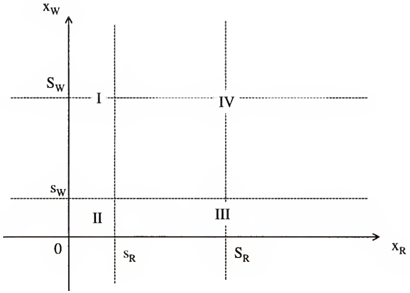


Figure 2.2: Possible regions for values of inventory level prior to making decision at the retailer (x_R) and the warehouse (x_W)

The uniqueness of s_R and s_W are guaranteed because $G(y_R)$ and $G(y_W)$ are strictly convex function of y_R and y_W.

To get more insight into this problem, we can divide the possible solutions of x_R and x_W into 4 regions according to predefined ordering rules. These regions are represented by Figure 2.2. One more assumption needs to be made at this point: the warehouse places and receives an order before the retailer does.

- If (x_R, x_W) falls in area I ($x_R \leq s_R,$ *and* $s_W < x_W \leq S_W$), the retailer will place an order of $S_R - x_R$ in order to bring the inventory level up to S_R. Unfortunately, sometimes the warehouse may not have enough inventories to fill the request. Thus, there are two likely events occurring under this area:

– If $x_W \geq (S_R - x_R)$; the quantity received is $S_R - x_R$. We will call this action ordering decision 1(a). The total expected cost associated with this decision can be expressed as

$$C_{1(a)}(x_R, x_W) = K_R - c_R x_R - c_W x_W + G(S_R, x_W) \tag{2.13}$$

– If $x_W < (S_R - x_R)$, then the retailer will have x_W after ordering rather than S_R. We will call this action ordering decision 1(b). The total expected cost associated with this decision can be expressed as

$$C_{1(b)}(x_R, x_W) = K_R - c_R x_R - c_W x_W + G(x_W, x_W) \tag{2.14}$$

- If (x_R, x_W) falls in area II ($x_R \leq s_R,\ and\ x_W \leq s_W$), the retailer and the warehouse will place an order of $S_R - x_R$ and $S_W - x_W$ in order to bring the inventory level up to S_R and S_W respectively. Thus, the amount to be received at the retailer is $S_R - x_R$, and at the warehouse is $S_W - x_W$. We will call this action ordering decision 2. The total expected cost associated with this decision can be expressed as

$$C_2(x_R, x_W) = K_R + K_W - c_R x_R - c_W x_W + G(S_R, S_W) \tag{2.15}$$

- If (x_R, x_W) falls in area III ($s_R < x_R \leq S_R,\ and\ x_W \leq s_W$), the warehouse will place an order of $S_W - x_W$ in order to bring the inventory level up to S_W. As mentioned above, the source has infinite supplies, so the warehouse receives inventories in an amount of $S_W - x_W$. We will call this action ordering decision

3. The total expected cost associated with this decision can be expressed as

$$C_3(x_R, x_W) = K_W - c_R x_R - c_W x_W + G(x_R, S_W) \tag{2.16}$$

- Finally, if (x_R, x_W) falls in area IV ($s_R < x_R,\ and\ s_W < x_W$), both warehouses will not order anything. We will call this action as ordering decision 4. The total expected cost associated with this decision can be expressed as

$$C_4(x_R, x_W) = -c_R x_R - c_W x_W + G(x_R, S_W) \tag{2.17}$$

2.5 An Example with Negative Exponential Demand Distribution

Suppose that the demand(D) for the commodity at the retailer has negative exponential distribution with mean $\frac{1}{\lambda}$. Let

$$\varphi_D(\xi) = \lambda e^{-\lambda\xi}, \tag{2.18}$$

$$\Phi_D(\xi) = 1 - e^{-\lambda\xi},\ \lambda > 0,\ 0 \le \xi \le \infty. \tag{2.19}$$

From equation (2.5), (2.6), (2.8), and (2.10), we can write $G(y_R)$ and $G(y_W)$ as follows:

$$G(y_R) = (h_R + p_R)\int_0^{y_R} (y_R - \xi)\varphi_D(\xi)\,d\xi + \frac{p_R}{\lambda} - (p_R - c_R)y_R \tag{2.20}$$

$$G(y_W) = (h_W + p_W)\int_0^{y_W} (y_W - \xi)\varphi_D(\xi)\,d\xi + \frac{p_W}{\lambda} - (p_W - c_W)y_W. \tag{2.21}$$

Then,

$$\frac{dG(y_R)}{dy_R} = 0 = (h_R + p_R)\Phi_D(y_R) - (p_R - c_R)$$

$$\frac{dG(y_W)}{dy_W} = 0 = (h_W + p_W)\Phi_D(y_W) - (p_W - c_W)$$

Hence, if $p_R > c_R$ and $p_W > c_W$, $S_R > 0$ and $S_W > 0$ can be obtained from

$$\Phi_D(S_R) = \frac{p_R - c_R}{h_R + p_R} \tag{2.22}$$

$$\Phi_D(S_W) = \frac{p_W - c_W}{h_W + p_W}. \tag{2.23}$$

Substituting (2.19) in (2.22) and (2.23), we obtain S_R and S_W as follows:

$$S_R = \frac{1}{\lambda} \ln \frac{h_R + p_R}{h_R + c_R}$$

$$S_W = \frac{1}{\lambda} \ln \frac{h_W + p_W}{h_W + c_W}$$

Now, for s_R and s_W, substitute (2.18) in (2.20) and (2.21):

$$G(y_R) = (h_R + p_R)(y_R + \frac{e^{-\lambda y_R}}{\lambda} - \frac{1}{\lambda}) + \frac{p_R}{\lambda} - (p_R - c_R)y_R \tag{2.24}$$

$$G(y_W) = (h_W + p_W)(y_W + \frac{e^{-\lambda y_W}}{\lambda} - \frac{1}{\lambda}) + \frac{p_W}{\lambda} - (p_W - c_W)y_W. \tag{2.25}$$

Substituting (2.24) in (2.11) and (2.25) in (2.12), we will obtain s_R and s_W by solving the following equations.

$$K_R = (h_R + p_R)[(s_R + \frac{e^{-\lambda s_R}}{\lambda}) - (S_R + \frac{e^{-\lambda S_R}}{\lambda})] - (p_R - c_R)(s_R - S_R)$$

$$K_W = (h_W + p_W)[(s_W + \frac{e^{-\lambda s_W}}{\lambda}) - (S_W + \frac{e^{-\lambda S_W}}{\lambda})] - (p_W - c_W)(s_W - S_W)$$

CHAPTER 3
THE LINEAR TRANSPORTATION COST MODEL

3.1 Introduction

Two major cost components in logistics are inventory and transportation costs. In general, the products or commodities are manufactured at the factory and kept in the warehouse before being shipped out to retailers or end customers. The transportation cost is incurred by moving commodities over space while the inventory cost may be incurred from holding excess commodities at the retailers. The transportation cost may be reduced by shipping commodities in larger shipment with less frequencies. However, the inventory holding cost may increase. Determining inventory and transportation decisions separately generally lead to a high system total cost. As a result, there is an increasing need of integrated models which coordinate inventory control and transportation planning aspects [12]. The objective is to determine an integrating decision so as to minimize the system inventory and transportation costs. The integrating decision results from the tradeoff between inventory cost and transportation costs.

This chapter develops a mathematical model to minimize inventory and transportation cost in multi-warehouse multi-retailer system with stochastic demand subject to finite capacity at warehouses. Initially, we investigate the model of linear transportation cost. Since the objective function is the sum of the inventory convex cost function

and the linear transportation cost, the problem becomes a nonlinear optimization problem. Silver and et al. [52] derive the solution of the similar problem with one resource through the use of Lagrange multipliers.

The chapter is organized as follows. A literature reviewed is presented in Section 3.2. Section 3.3 gives details of the problem, introduces necessary assumptions and notation, and presents the mathematical formulation. The Lagrange multiplier method is described in Section 3.4. Then the analysis of the two-warehouse two-retailer, single warehouse multi-retailer, and multi-warehouse multi-retailer system is shown in Section 3.5, 3.6, and 3.7 respectively. Finally, we conclude the chapter in Section 3.8.

3.2 Literature Review

The purpose of this section is to survey the broad variety of integrated inventory and transportation models. As reviewed in the literature, the transportation cost is defined in different circumstances. It may refer to a function of cost per trip [55, 17], cost per shipping unit (embedded in inventory model), or vehicle routing cost [23]. The inventory cost is defined based on deterministic or stochastic demand models. In the deterministic model, the inventory cost refers to the holding cost charged against excess inventory in the system. In the stochastic demand models, in addition to the holding cost, there is a shortage or penalty cost. It is charged for not being able to complete the demand.

Early studies in an integrated inventory-transportation system is done by Federgruen and Zipkin [23]. They develop a stochastic model which combine the inventory allocation problem and the standard vehicle routing problem. An inventory allocation problem determines optimal inventory replenishing policies among several locations (e.g., retailers

or end customers) by sharing a limited resource (e.g., finite capacity warehouse or fleet vehicles). The vehicle routing problem (*VRP*) determines a delivery plan (vehicle routes) for a fleet of vehicles at minimum system total cost. The authors consider a single-period system of one central depot multiple locations with stochastic demand. They manipulate the model in order to partially decompose it into an inventory problem and several VRP subproblems. The interchange heuristic, a well known solution method for the deterministic VRP, is modified to manage their stochastic model.

Recently, Bertazzi and Speranza [12] have surveyed the deterministic models and algorithms for the minimization of inventory and transportation during the last two decades. The models and algorithms are classified in continuous and discrete models. The reviewed models are considered under the following five basic types of logistics networks: the single link case (simplest network), the sequence of link case, the one origin-multiple destinations case, the multiple origins-one destination case, and the multiple origins-multiple destinations case. Most of literature in the continuous models are 'succinct modeling'. The results in many of these models are on the basis of the economic ordering quantity (*EOQ*). One disadvantage of the continuous time model is the solution may not be realistic from the practical point of view. The aim of Anily and Federgruen [3] is to determine the inventory replenishing policy which is incorporated with the transportation plan over the infinite horizon. They develop a single-item, deterministic model in one origin-multiple destinations network. The depot (an origin) is a transshipment point (i.e., there is no inventory kept at the depot). They propose a heuristic exploiting the regional partitioning strategy to cluster the destinations (retailers) into regions. It is possible that the same retailer belongs to several regions. That

means a retailer can be assigned to different routes simultaneously. Once the regions are established, an efficient route within each region is determined. The derived lower bound and upper bound are compared to show that these bounds become asymptotically tight when the number of destinations approaches infinity. Later in Anily and Federgruen [4] this model is extended to coordinate the inventory at the depot level. The inventory cost is computed by a more complicated function.

The single link case (one origin one destination) discrete time deterministic model is considered in Speranza and Ukovich [55]. The objective is to determine the trip frequencies in a multi-product system with a given set of feasible frequencies. That means only one frequency is assigned to each product. In their model, several products have to be shipped by vehicles of given capacity on a common link from an original node to a destination node. Four different situations are analyzed based on the shipping policies and consolidating rules. There are two schemes in shipping policies that each product can follow. It can be either shipped in only one frequency or partially shipped with multiple frequencies. The way that products are loaded on the vehicles is determined by the consolidating rules. It can be either frequency or time consolidation. Each situation is formulated in a mixed integer linear programming model. They have proven that the model with multiple frequencies and time consolidation is equivalent to the model with multiple frequencies and frequency consolidation. The numerical example shows that the best scheme is allowing products to be partially shipped (multiple frequencies) and share the same truck with the same frequency product.

Burns et al. [17] use an analytical approach to minimize the inventory and transportation costs in one origin-multiple destinations network. Their model considers only

one product with deterministic demand. Two distribution strategies are developed, evaluated and compared. The first strategy is direct shipping (separate loads directly to each customer). The second strategy is peddling (more than one customer in one load). In the peddling strategy, customers are divided into delivery regions. Each truckload is delivered to customers within the same region. The transportation cost for the peddling involves three stages: *line-haul* (from the supplier to the first customer stop), *local delivery* (from the first customer stop to the last customer stop), and *back-haul* (from last customer stop to the supplier). While direct shipping uses the EOQ-based optimal shipment size, the peddling shipment size is a full truck. In addition, the results denote that the peddling is superior to the direct shipping with an increase of the following elements: item value, distance from supplier, customer density in the region, and inventory holding cost.

3.3 Problem Descriptions and Model Formulation

This section describes the capacitated inventory and transportation problem in a multi-warehouse multi-retailer system with stochastic demand.

First let us revisit the single-period, single-commodity uncapacitated inventory problem with stochastic demand. The decision to be made is how much to order at the beginning of the period while the expected total cost is minimizes. The expected total cost is comprised of two components: holding cost and penalty or shortage cost. The holding cost accrues as a result of having positive inventory while the penalty cost or shortage cost is charged against a negative level of inventory (unable to satisfy demands) at the end of period. Since the objective function is convex (shown in previous chapter) and

there are unlimited resources, then there is a unique optimal solution. Let x^* denote the optimal solution. Then $x^*$is derived from

$$\Phi(x^*) = \frac{p}{h+p}$$

where p is a cost per unit of unsatisfied demand and h is a cost per unit of having positive inventory at the end of the period.

As mentioned in Chapter 1, in a transportation problem, a set of supply nodes serves a set of demand nodes. The decision to be made is how many units to be shipped on each arc between supply nodes and demand nodes in order to minimize the total transportation cost. This decision must not exceed the limited resources available at the supply side and must satisfy the amount needed at the demand side.

Now consider a system consisting of m warehouses supplying n retailers (see Figure 3.1). Let i index the warehouses in the set $I = \{1, \ldots, m\}$ and let j index the retailers in the set $J = \{1, \ldots, n\}$. Random demands take place at retailers. Warehouses have finite capacity. The shipping flow can be induced from any warehouse to any retailer. When the inventory model is extended to embrace transportation cost, the optimal solution is not only influenced by the holding and penalty cost but also the unit transportation cost.

3.3.1 Assumptions and Notation

This subsection states the necessary assumptions and notation.

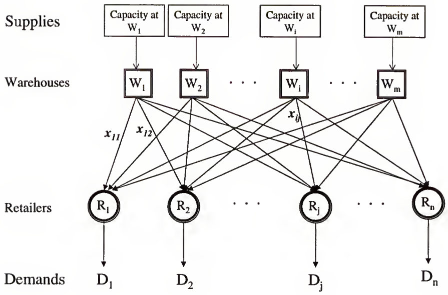


Figure 3.1: A capacity constrained multi-warehouse multi-retailer system

Assumptions

1. The capacitated inventory and transportation problem is formulated as a discrete-time, single period, multi-warehouse, multi-retailer problem.
2. The warehouses hold single-type commodities.
3. There is no transshipment between warehouses or retailers.
4. The shipping decision is made at the beginning of the period.
5. The decision specifies the shipping amount and its associated shipping arc.
6. Supply at warehouses is limited.
7. Demands at retailers are stochastic with known means and known variances, and independently distributed. Unsatisfied demands are completely lost sales.
8. At the beginning of the period, the inventory level at all retailers is zero.

9. The total cost includes inventory cost (holding/penalty cost) and transportation cost.
10. Inventory costs occur at retailers.
11. The transportation cost can have a linear form (considered in this chapter) and a fixed charge form (considered in the subsequent chapters).

Notation

The following notation is used in the rest of this dissertation.

I	a set of warehouses
J	a set of retailers
m	number of warehouses
n	number of retailers
W_i	warehouse i in set I
R_j	retailer j in set J
x_{ij}	shipping amount (decision variable) from W_i to R_j
$\varphi_{D_j}(\xi_j)$	demand probability density function at R_j
$\Phi_{D_j}(\xi_j)$	cumulative demand distribution function at R_j
	where $\Phi_{D_j}(\xi_j) = \int_0^{\xi_j} \varphi_{D_j}(u)du$
$\bar{D}_j$	mean of demand at R_j
p_j	penalty or shortage cost at R_j
h_j	holding cost at R_j
C_i	capacity at W_i

3.3.2 Model Formulation

The involved costs

As mentioned earlier, the expected total cost includes the inventory cost and the transportation cost. Let ICj denote the inventory cost at R_j, and T_{ij} denote the transportation cost from W_i to R_j. It should be noted that the inventory cost is evaluated from the the total amount received at each retailer which is supplied from at least one warehouse. Thus the inventory cost is a function of $\sum_{i=0}^{m} x_{ij}$. That means it is not separable for each arc ij (see Equation (3.1)). The expected inventory cost for each retailer is expressed as follows.

$$ICj(\sum_{i=0}^{m} x_{ij}) =$$
$$h_j \int_0^{\sum_{i=1}^{m} x_{ij}} (\sum_{i=1}^{m} x_{ij} - \xi_j)\varphi_{D_j}(\xi_j)d\xi_j + p_j \int_{\sum_{i=1}^{m} x_{ij}}^{\infty} (\xi_j - \sum_{i=1}^{m} x_{ij})\varphi_{D_j}(\xi_j)d\xi_j \quad (3.1)$$

While the inventory cost is a convex function, the transportation cost can take any form, eg. linear, fixed charge, etc. Later in this chapter, we consider the simplest case, a linear transportation cost. Now let $TC(x_{ij})$ denote the expected total cost of the system. Then we can write $TC(x_{ij})$ as follows.

$$TC(x_{ij}) = \sum_{j=0}^{n} \left(ICj(\sum_{i=0}^{m} x_{ij}) + \sum_{i=0}^{m} T_{ij}(x_{ij}) \right) \quad (3.2)$$

Substitute Equation (3.1) in Equation (3.2) we obtain

$$\begin{aligned} TC(x_{ij}) &= \sum_{j=0}^{n}\left(h_j \int_0^{\sum_{i=1}^{m} x_{ij}} (\sum_{i=1}^{m} x_{ij} - \xi_j)\varphi_{D_j}(\xi_j)d\xi_j \right. \\ &\quad \left. + p_j \int_{\sum_{i=1}^{m} x_{ij}}^{\infty} (\xi_j - \sum_{i=1}^{m} x_{ij})\varphi_{D_j}(\xi_j)d\xi_j \right) \\ &\quad + \sum_{j=0}^{n}\sum_{i=0}^{m} T_{ij}(x_{ij}) \end{aligned}$$

Model formulation

The objective is to find the optimal integrating inventory and transportation decision that minimizes the expected total cost.

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} \left(h_j \int_0^{\sum_{i=1}^{m} x_{ij}} (\sum_{i=1}^{m} x_{ij} - \xi_j)\varphi_{D_j}(\xi_j)d\xi_j \right. \\ & \quad \left. + p_j \int_{\sum_{i=1}^{m} x_{ij}}^{\infty} (\xi_j - \sum_{i=1}^{m} x_{ij})\varphi_{D_j}(\xi_j)d\xi_j \right) \\ & + \sum_{j=1}^{n}\sum_{i=1}^{m} T_{ij}(x_{ij}) \\ s.t. \quad & \sum_{j=1}^{n} x_{ij} \leq C_i \qquad \forall i \\ & x_{ij} \geq 0 \qquad \forall i,j \end{aligned} \tag{3.3}$$

There are two sets of constraints. The first set is the capacity constraints. The total amount shipped from W_i to any retailers must be at most the available capacity at W_i. The second set is nonnegativity constraints.

A linear transportation cost model formulation

In this chapter, we consider a linear transportation cost. Let v_{ij} denote the cost associated with moving one unit of commodity from the W_i to the R_j. Then $T_{ij}(x_{ij})$ is

expressed as

$$T_{ij}(x_{ij}) = v_{ij}x_{ij}. \tag{3.4}$$

Substitute Equation (3.4) in the problem (3.3), the formulation becomes

$$\begin{aligned}
\min \quad & \sum_{j=1}^{n} \Bigg(h_j \int_0^{\sum_{i=1}^{m} x_{ij}} (\sum_{i=1}^{m} x_{ij} - \xi_j)\varphi_{D_j}(\xi_j)d\xi_j \\
& \qquad + p_j \int_{\sum_{i=1}^{m} x_{ij}}^{\infty} (\xi_j - \sum_{i=1}^{m} x_{ij})\varphi_{D_j}(\xi_j)d\xi_j \Bigg) \\
& + \sum_{j=1}^{n}\sum_{i=1}^{m}(v_{ij}x_{ij}) \\
s.t. \quad & \sum_{j=1}^{n} x_{ij} \ \leq \ C_i \qquad \forall i \\
& x_{ij} \qquad \geq \ 0 \qquad \forall i,j
\end{aligned} \tag{3.5}$$

3.4 Lagrange Multiplier Method for Inequality Constraints Problem

This section describes the Lagrange multiplier method to solve a nonlinear programming problem with inequality constraints. The method is summarized from [13, 14]. Consider a problem involving inequality constraints (ICP)

$$\begin{aligned}
\min \quad & f(x) \\
s.t. \quad & g_1 \ \leq \ 0, \ \ldots, \ g_m \ \leq \ 0
\end{aligned}$$

Assume that $f : \Re^n \rightarrow \Re$, $g : \Re^n \rightarrow \Re^m$ are continuously differentiable functions. Let us introduce additional variables $\alpha_1, \ldots, \alpha_m$. We can convert the ICP above to the equality constraint problem as follows:

$$\begin{aligned}
\min \quad & f(x) \\
s.t. \quad & g_1 + \alpha_1^2 \ = \ 0, \ \ldots, \ g_m + \alpha_m^2 \ = \ 0
\end{aligned}$$

Let x^* be a local minimum for our original problem (ICP). Then (x^*, α^*) is a local minimum for converted problem, where $\alpha^* = (\alpha_1^*, \dots, \alpha_m^*)$,

$$\alpha_j^* = (-g_j(x^*))^{1/2} \qquad \text{for } j = 1, \dots, m.$$

For all j's, then there exist unique Lagrange multipliers λ_j^* such that

$$\nabla f(x^*) + \sum_{j=1}^{m} \lambda_j \nabla g_j(x^*) = 0,$$

$$g_j + \alpha_j^2 = 0$$

$$2\lambda_j \alpha_j = 0$$

3.5 The Two-Warehouse Two-Retailer Problem

This section analyzes a simple system consisting of two warehouses and two retailers. We can formulate the problem as follows:

$$\begin{aligned}
\min \quad & h_1 \int_0^{x_{11}+x_{21}} \left((x_{11}+x_{21}) - \xi_1\right) \varphi_{D_1}(\xi_1) d\xi_1 \\
& + p_1 \int_{x_{11}+x_{21}}^{\infty} \left(\xi_1 - (x_{11}+x_{12})\right) \varphi_{D_1}(\xi_1) d\xi_1 \\
& + c_{11}x_{11} + c_{21}x_{21} \\
& + h_2 \int_0^{x_{12}+x_{22}} \left((x_{12}+x_{22}) - \xi_2\right) \varphi_{D_2}(\xi_2) d\xi_2 \\
& + p_2 \int_{x_{12}+x_{22}}^{\infty} \left(\xi_2 - (x_{12}+x_{22})\right) \varphi_{D_2}(\xi_2) d\xi_2 \\
& + v_{12}x_{12} + v_{22}x_{22} \\
s.t. \quad & x_{11} + x_{12} \le C_1 \\
& x_{21} + x_{22} \le C_2 \\
& x_{11}, x_{12}, x_{21}, x_{22} \ge 0
\end{aligned}$$

Converting the two inequality constraints to the equality case, we obtain

$$\begin{aligned} x_{11} + x_{12} + \alpha_1^2 &= C_1 \\ x_{21} + x_{22} + \alpha_2^2 &= C_2 \end{aligned}$$

The Lagrangian function of the problem can be expressed as

$$\begin{aligned} &F(x_{11}, x_{12}, x_{21}, x_{22}, \lambda_1, \lambda_2, \alpha_1, \alpha_2) = \\ &\quad h_1 \int_0^{x_{11}+x_{21}} ((x_{11} + x_{21}) - \xi_1)\varphi_{D_1}(\xi_1)d\xi_1 + p_1 \int_{x_{11}+x_{21}}^{\infty} (\xi_1 - (x_{11} + x_{12}))\varphi_{D_1}(\xi_1)d\xi_1 \\ &\quad +v_{11}x_{11} + v_{21}x_{21} + h_2 \int_0^{x_{12}+x_{22}} ((x_{12} + x_{22}) - \xi_2)\varphi_{D_2}(\xi_2)d\xi_2 \\ &\quad +p_2 \int_{x_{12}+x_{22}}^{\infty} (\xi_2 - (x_{12} + x_{22}))\varphi_{D_2}(\xi_2)d\xi_2 + v_{12}x_{12} + v_{22}x_{22} \\ &\quad +\lambda_1(x_{11} + x_{12} + \alpha_1^2 - C_1) + \lambda_2(x_{21} + x_{22} + \alpha_2^2 - C_2) \end{aligned} \tag{3.6}$$

By rearranging terms and using integration by parts technique, Equation (3.6) is equivalent to

$$\begin{aligned} &F(x_{11}, x_{12}, x_{21}, x_{22}, \lambda_1, \lambda_2, \alpha_1, \alpha_2) = \\ &\quad (h_1 + p_1) \int_0^{x_{11}+x_{21}} \Phi_{D_1}(\xi_1)d\xi_1 + p_1\bar{D}_1 - (p_1 - v_{11})x_{11} - (p_1 - v_{21})x_{21} \\ &\quad +(h_2 + p_2) \int_0^{x_{12}+x_{22}} \Phi_{D_2}(\xi_2)d\xi_2 + p_2\bar{D}_2 - (p_2 - v_{12})x_{12} - (p_2 - v_{22})x_{22} \\ &\quad +\lambda_1(x_{11} + x_{12} + \alpha_1^2 - C_1) + \lambda_2(x_{21} + x_{22} + \alpha_2^2 - C_2) \end{aligned} \tag{3.7}$$

Partially differentiating the Lagrangian function (3.7) with respect to all variables, the necessary conditions become

$$
\begin{aligned}
(h_1+p_1)\Phi_{D_1}(x_{11}^*+x_{21}^*)-(p_1-v_{11})+\lambda_1^* &= 0\\
(h_1+p_1)\Phi_{D_1}(x_{11}^*+x_{21}^*)-(p_1-v_{21})+\lambda_2^* &= 0\\
(h_2+p_2)\Phi_{D_2}(x_{12}^*+x_{22}^*)-(p_2-v_{12})+\lambda_1^* &= 0\\
(h_2+p_2)\Phi_{D_2}(x_{12}^*+x_{22}^*)-(p_2-v_{22})+\lambda_2^* &= 0\\
x_{11}^*+x_{12}^*+\alpha_1^{*2}-C_1 &= 0\\
x_{21}^*+x_{22}^*+\alpha_2^{*2}-C_2 &= 0\\
2\lambda_1^*\alpha_1^* &= 0\\
2\lambda_2^*\alpha_2^* &= 0
\end{aligned}
$$

where (x_{ij}^*, α_i^*) is a local minimum.

The Lagrange multipliers, λ_1 and λ_2, can be either zero or a positive value while α_1 and α_2 can be either zero or nonzero. It should be noted that for any i, at least λ_i or α_i has to be zero. For each warehouse, there are three possible cases of λ_i and α_i combination. Thus, there are nine cases for the two-warehouse problem to be considered. The necessary conditions for these cases are summarized in Table 3.1. The optimal solution satisfies conditions in one of these cases. Suppose that the optimal λ_1^*, λ_2^* ,α_1^*, and α_2^* are found, we obtain the solution from solving the following:

$$x_{11}^*+x_{21}^*=\Phi_{D_1}^{-1}\left(\frac{p_1-v_{11}-\lambda_1^*}{h_1+p_1}\right)$$

$$x_{11}^*+x_{21}^*=\Phi_{D_1}^{-1}\left(\frac{p_1-v_{21}-\lambda_2^*}{h_1+p_1}\right)$$

Table 3.1: Summary of necessary conditions for an optimal solution to a two-warehouse two-retailer problem

Case	λ_1	α_1	λ_2	α_2	*Conditions*
1	0	0	0	0	$v_{11} = v_{21}; v_{12} = v_{22};$ $x_{11} + x_{21} + x_{12} + x_{22} = C_1 + C_2$
2	0	0	+	0	$v_{11} - v_{21} = v_{12} - v_{22}; v_{11} > v_{21}; v_{12} > v_{22};$ $x_{11} + x_{21} + x_{12} + x_{22} = C_1 + C_2$
3	0	0	0	-/+	$v_{11} = v_{21}; v_{12} = v_{22};$ $x_{11} + x_{12} = C_1; x_{21} + x_{22} < C_2$
4	0	-/+	0	0	$v_{11} = v_{21}; v_{12} = v_{22};$ $x_{11} + x_{12} < C_1; x_{21} + x_{22} = C_2$
5	+	0	0	0	$v_{21} - v_{11} = v_{22} - v_{12}; v_{21} > v_{11}; v_{22} > v_{12};$ $x_{11} + x_{21} + x_{12} + x_{22} = C_1 + C_2$
6	0	-/+	0	-/+	$v_{11} = v_{21}; v_{12} = v_{22}; v_{11} > v_{21};$ $x_{11} + x_{21} < C_1;\ x_{12} + x_{22} < C_2$
7	0	-/+	+	0	$v_{11} - v_{21} = v_{12} - v_{22}; v_{11} > v_{21}; v_{12} > v_{22};$ $x_{11} + x_{21} < C_1; x_{12} + x_{22} = C_2$
8	+	0	0	-/+	$v_{21} - v_{11} = v_{22} - v_{12}; v_{21} > v_{11}; v_{22} > v_{12};$ $x_{11} + x_{21} = C_1; x_{12} + x_{22} < C_2$
9	+	0	+	0	$v_{21} - v_{11} = v_{22} - v_{12};$ $x_{11} + x_{21} + x_{12} + x_{22} = C_1 + C_2$

$$x_{12}^* + x_{22}^* = \Phi_{D_2}^{-1}\left(\frac{p_2 - v_{12} - \lambda_1^*}{h_2 + p_2}\right)$$

$$x_{12}^* + x_{22}^* = \Phi_{D_2}^{-1}\left(\frac{p_2 - v_{22} - \lambda_2^*}{h_2 + p_2}\right)$$

$$x_{11}^* + x_{12}^* + \alpha_1^{*2} = C_1$$

$$x_{21}^* + x_{22}^* + \alpha_2^{*2} = C_2$$

3.5.1 Numerical Example

Now suppose that R_1 and R_2 have negative exponential demand distribution with mean $\bar{D}_1 = 100$ units/month and $\bar{D}_2 = 50$ units/month. The ratios of holding and penalty cost at R_1 and R_2 are given as $\frac{p_1}{h_1} = 20$ and $\frac{p_2}{h_2} = 25$. The holding cost is

evaluated by $h = iV$, where i is interest rate (cost of capital) and V is a value of commodity per unit. Thus, $h_1 = i_1 V$ and $h_2 = i_2 V$, where $i_1 = 20$ and $i_2 = 15\$/\$(year)$. The transportation cost per unit is given as follows:

$$v_{11} = 10 \text{ \$/unit}; \quad v_{12} = 20 \text{ \$/unit}; \quad v_{21} = 5 \text{ \$/unit}; \quad v_{22} = 15 \text{ \$/unit}$$

Finally, the capacity at both warehouses are equal to 100 units. From the given data above, the Lagrangian function becomes

$$\begin{aligned}
&F(x_{11}, x_{12}, x_{21}, x_{22}, \lambda_1, \lambda_2, \alpha_1, \alpha_2) = \\
&\quad (69.9)\int_0^{x_{11}+x_{21}} \Phi_{D_1}(\xi_1)d\xi_1 + (66.6)(100) - (66.6)(x_{11} + x_{21}) + (10)x_{11} + (5)x_{21} + \\
&\quad (39)\int_0^{x_{12}+x_{22}} \Phi_{D_2}(\xi_2)d\xi_2 + (37.5)(50) - (37.5)(x_{12} + x_{22}) + (20)x_{12} + (15)x_{22} + \\
&\quad \lambda_1(x_{11} + x_{12} + \alpha_1^2 - 100) + \lambda_2(x_{21} + x_{22} + \alpha_2^2 - 100)
\end{aligned}$$

There exist Lagrange multipliers $\lambda_1^* = 0$ and $\lambda_2^* = 5$ such that

$$\begin{aligned}
(69.9)\Phi_{D_1}(x_{11}^* + x_{21}^*) - 56.6 + \lambda_1^* &= 0 \\
(69.9)\Phi_{D_1}(x_{11}^* + x_{21}^*) - 61.6 + \lambda_2^* &= 0 \\
(39)\Phi_{D_2}(x_{12}^* + x_{22}^*) - 17.5 + \lambda_1^* &= 0 \\
(39)\Phi_{D_2}(x_{12}^* + x_{22}^*) - 22.5 + \lambda_2^* &= 0 \\
x_{11}^* + x_{12}^* + \alpha_1^{*2} - 100 &= 0 \\
x_{21}^* + x_{22}^* + \alpha_2^{*2} - 100 &= 0 \\
2\lambda_1^*\alpha_1^* &= 0 \\
2\lambda_2^*\alpha_2^* &= 0
\end{aligned}$$

Then we obtain the following solutions:

$$
\begin{aligned}
x_{11}^* + x_{21}^* &= 170 \\
x_{12}^* + x_{22}^* &= 30 \\
x_{11}^* + x_{12}^* &= 100 \\
x_{21}^* + x_{22}^* &= 100
\end{aligned}
$$

Solving this system of linear equations, we obtain many values of x_{11}^*, x_{12}^*, x_{21}^*, x_{22}^* but they all give the same objective value of \$4,348.14. The example of solutions are ($x_{11}^* = 70$, $x_{12}^* = 30$, $x_{21}^* = 100$, $x_{22}^* = 0$), ($x_{11}^* = 80$, $x_{12}^* = 20$, $x_{21}^* = 90$, $x_{22}^* = 10$), etc.

3.6 The Single-Warehouse Multi-Retailer Problem

This section considers a system consisting of one warehouse and n retailers. The problem can be formulated as follows:

$$
\begin{aligned}
\min \quad & (h_1 + p_1) \int_0^{x_1} \Phi_{D_1}(\xi_1) d\xi_1 + p_1 \bar{D}_1 - p_1 x_1 + v_1 x_1 + \\
& \vdots \\
& (h_n + p_n) \int_0^{x_n} \Phi_{D_n}(\xi_n) d\xi_n + p_n \bar{D}_n - p_n x_n + v_n x_n \\
s.t. \quad & x_1 + \ldots + x_n \leq I \\
& x_j \geq 0 \quad for\, j = 1, \ldots, n
\end{aligned}
$$

Note that since we have only one warehouse, the subscript is suppressed to j (index on *retailer*). Converting the capacity inequality constraints to the equality case, we obtain

$$x_1 + \ldots + x_n + \alpha^2 = C$$

The Lagrangian function of the problem can be expressed as

$$
\begin{aligned}
&F(x_1, \ldots, x_n, \lambda, \alpha) = \\
&\quad (h_1 + p_1) \int_0^{x_1} \Phi_{D_1}(\xi_1) d\xi_1 + p_1 \bar{D}_1 - p_1 x_1 + v_1 x_1 + \\
&\quad \vdots \\
&\quad (h_n + p_n) \int_0^{x_n} \Phi_{D_n}(\xi_n) d\xi_n + p_n \bar{D}_n - p_n x_n + v_n x_n + \\
&\quad \lambda(x_1 + \ldots + x_n + \alpha^2 - I)
\end{aligned}
$$

The necessary conditions become

$$
\begin{aligned}
(h_1 + p_1)\Phi_{D_1}(x_1^*) - (p_1 - v_1) + \lambda^* &= 0 \\
&\vdots \\
(h_n + p_n)\Phi_{D_n}(x_n^*) - (p_n - v_n) + \lambda^* &= 0 \\
x_{11}^* + x_{12}^* + \alpha_1^{*2} - C_1 &= 0 \\
2\lambda^* \alpha^* &= 0
\end{aligned}
$$

Then the solution is

$$
\begin{aligned}
x_j^* &= \Phi_{D_1}^{-1}\left(\frac{p_j - v_j - \lambda^*}{h_j + p_j}\right) \quad \text{for } j = 1, \ldots, n \\
\lambda^* &= \arg \min_{\lambda \geq 0} \left\{ \sum_{j=1}^{n} \Phi_{D_1}^{-1}\left(\frac{p_j - v_j - \lambda^*}{h_j + p_j}\right) \leq C \right\}
\end{aligned}
$$

Table 3.2 summarizes the possible values for λ and α and necessary conditions for an optimal solution.

Table 3.2: Summary of necessary conditions for an optimal solution to a single warehouse n retailers problem

Case	λ	α	*Conditions*
1	0	0	$\Phi_{D_j}(x_j^*) = \frac{p_j - v_j}{h_j + p_j}$; $\sum_j x_j = I$
2	0	-/+	$\Phi_{D_j}(x_j^*) = \frac{p_j - v_j}{h_j + p_j}$; $\sum_j x_j < I$
3	+	0	$\Phi_{D_j}(x_j^*) = \frac{p_j - v_j - \lambda}{h_j + p_j}$; $\sum_j x_j = I$

3.7 The Multi-Warehouse Multi-Retailer Problem

This section analyzes a general case when the system is consisting of m warehouses and n retailers. The formulation is shown below.

$$\begin{aligned}
\min \quad & \sum_{j=1}^{n} \left\{ (h_i + p_i) \int_0^{\sum_{i=0}^{m} x_{ij}} \Phi_{D_j}(\xi_j) d\xi_j + p_j \bar{D}_j - \sum_{i=1}^{m} p_j x_{ij} \right\} + \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} x_{ij} \\
s.t. \quad & \sum_{j=1}^{n} x_{ij} \;\leq\; C_i \qquad \forall i \\
& x_{ij} \;\geq\; 0 \qquad \forall i, j
\end{aligned}$$

Converting m inequality constraints to the equality ones, the Lagrangian function can be expressed as follows:

$$\begin{aligned}
& F(x_{ij}, \lambda_i, \alpha_i) = \\
& \quad \sum_{j=1}^{n} \left\{ (h_j + p_j) \int_0^{\sum_{i=0}^{m} x_{ij}} \Phi_{D_j}(\xi_j) d\xi_j + p_j \bar{D}_j - \sum_{i=1}^{m} p_j x_{ij} \right\} + \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} x_{ij} \\
& \quad + \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} x_{ij} + \alpha_i^2 - C_i \right)
\end{aligned}$$

Equivalently,

$$F(x_{ij}, \lambda_i, \alpha_i) =$$
$$\sum_{j=1}^{n} \left\{ (h_j + p_j) \int_0^{\sum_{i=0}^{m} x_{ij}} \Phi_{D_j}(\xi_j) d\xi_j + p_j \bar{D}_j - (p_j - v_{ij} - \lambda_i) \sum_{i=1}^{m} x_{ij} \right\}$$
$$+ \sum_{i=1}^{m} \lambda_i (\alpha_i^2 - C_i)$$

Thus the necessary conditions become

$$(h_j + p_j)\Phi_{D_j}(\sum_{i=1}^{m} x_{ij}^*) - (p_j - v_{ij} - \lambda_i^*) = 0 \ \forall i, j \tag{3.8}$$
$$\sum_{j=1}^{n} x_{ij}^* + \alpha_i^{*2} - C_i = 0 \ \forall i \tag{3.9}$$
$$2\lambda_i^* \alpha_i^* = 0 \ \forall i \tag{3.10}$$

As mentioned earlier, for any i at least λ_i or α_i has to be zero. For each warehouse, (λ_i, α_i) can be one of three cases: (0, 0), (0, +/-), or (+, 0). Therefore for m warehouses, the total sets of λ_i and α_i combination are 3^m! It is increasing exponentially with the number of warehouses.

Suppose that the optimal Lagrange multipliers, λ_i^*'s and α_i^*'s are known, then the solution can be obtained from

$$\sum_{i=1}^{m} x_{ij} = \Phi_{D_j}^{-1} \left(\frac{p_j - v_{ij} - \lambda_i^*}{h_j + p_j} \right) \tag{3.11}$$
$$\sum_{j=1}^{n} x_{ij}^* + \alpha_i^{*2} - C_i = 0 \ \forall i \tag{3.12}$$

However, obtaining the λ_i^* and α_i^* is not trivial. Thus, analytically solving the multi-retailer multi-retailer system is very tedious work.

3.8 Conclusions

This chapter develops the mathematical model to minimize the sum of inventory (holding and penalty costs) and transportation costs for a multi-warehouse multi-retailer system with stochastic demand. The analysis initially considers the linear transportation cost.

We first investigate a simple model with two warehouses and two retailers in the system. The Lagrange multiplier method is employed to obtain the optimal solution. The optimal total amount received at the retailer can be derived from the 'newsboy' formula assuming that the optimal Lagrange multipliers are known. Obtaining the optimal multipliers is complicated. Thus, we derive a set of feasible multipliers and its necessary conditions to ensure optimality. Then we apply the same technique to a single warehouse multi-retailer system. Lastly, we extend the result to the m-warehouse n-retailer system. For this system, the possible values of multipliers are increasing exponentially with the number of warehouses in the system. Thus, the Lagrange multiplier method is not efficient for solving large problems. In a later chapter, we develop a more powerful method which can efficiently and optimally solve the linear transportation cost problem.

CHAPTER 4
THE FIXED CHARGE TRANSPORTATION COST MODEL

4.1 Introduction

The previous chapter analyzes the case where we have a linear transportation cost. However, this linear type of cost structure is not practical. In this chapter we consider a more complex cost structure, a fixed charge transportation cost function, which exploits economies of scale. The fixed charge cost function, which has a discontinuity at the origin, is a special case of concave cost structure. Thus, the objective function has a convex (inventory cost) and a concave component (fixed charge transportation cost) which makes the problem nonconvex optimization problem.

As mentioned earlier, the fixed charge problem is a special case of a concave minimization problem. The early research in minimum concave cost network flows problem was done by Zangwill [60]. One approach for this class of problem is an enumerative method by ranking the extreme points [46]. This method exploits the concave function property that every local and global solution occurs at the extreme point of the feasible domain [48, 49]. Implementing this approach to a large scale problem is not practical because of the exhaustive use of the computational resource.

The most utilized approach for the fixed charge problem is branch and bound technique. An exact solution is found by transforming the fixed charge problem to a zero-one mixed integer programming (MIP). The branch and bound method finds an exact solution but it takes a lot of computational efforts.

Alternatively, many studies focus on developing heuristics to estimate the optimal solution. One of the most efficient heuristic is the Dynamic Slope Scaling Procedure (DSSP) developed by Kim and Pardalos [34, 35]. This heuristic is tested on a wide range of network structured problems. The reported computational results show that the DSSP obtains solutions optimally or near optimality within reasonable time. The heuristic can approximate the optimal solution for problems as large as 10000 arcs in a little more than one minute of CPU time.

The material of this chapter is organized along the following sequence. Section 3.2 approximates the inventory cost component using Monte Carlo method, and incorporates a fixed charge transportation cost. Section 3.3 discusses the DSSP to estimate the optimal of an approximated problem. Section 3.4 focuses on the computational experiment. The conclusions are presented in Section 3.5.

4.2 Model Formulation

4.2.1 Approximated Problem

Recall the integrated inventory and transportation problem formulation.

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} \Big(h_j \int_0^{\sum_{i=1}^m x_{ij}} \big(\sum_{i=1}^{m} x_{ij} - \xi_j\big)\varphi_{D_j}(\xi_j)d\xi_j + \\
& \qquad p_j \int_{\sum_{i=1}^m x_{ij}}^{\infty} \big(\xi_j - \sum_{i=1}^{m} x_{ij}\big)\varphi_{D_j}(\xi_j)d\xi_j\Big) + \\
& \sum_{j=1}^{n}\sum_{i=1}^{m} T_{ij}(x_{ij}) \\
s.t. \quad & \sum_{j=1}^{n} x_{ij} \ \leq \ C_i \qquad \forall i \\
& x_{ij} \ \geq \ 0 \qquad \forall i, j
\end{aligned}
$$

Algorithm IT-1

1. Generate U from $Uniform(0, 1)$
2. $X \leftarrow F_X^{-1}(U)$
3. Deliver X

Figure 4.1: Inverse Transformation (IT) algorithm

The most complicated component in the objective function above is the integrand of inventory cost component. In order to eliminate the integration while trying to maintain the randomness property of the demand function, we would estimate the integral by using numerical integration of Monte Carlo method. The Monte Carlo method changes the integration problem into a statistical estimation problem. In the changing process, first we follow the inverse probability integral transformation method to get sample points. Then the integral is estimated by the sample mean. Figure 4.1 presents the inverse transform method, where X is a random variable with cumulative probability distribution function (c.d.f) $F_X(x)$ [50]. The algorithm needs one random number to generate a sample point. Then this random number is transformed to a sample point through the inverse function. By $IT-1$ algorithm, we can randomly generate a set of demand points rather than using demand distribution function. Then we can estimate the integral with the mean of these generated demand points.

Let ξ_j^s be a demand random sample. Following $IT-1$ algorithm, we can obtain ξ_j^s from the equation below.

$$\xi_j^s = \Phi^{-1}(U)$$

where $\Phi(\bullet)$ is a demand distribution function, and U is a random number from Uniform(0,1). The index s indicates the s^{th} random sample generated or we call it scenario

for the rest of this chapter. Now, let S be a total number of scenarios generated to estimate the integral of inventory cost component. We substitute the inventory cost component with the integral estimating term, which is the sample mean of scenarios. The inventory holding cost term

$$h_j \int_0^{\sum_{i=1}^m x_{ij}} (\sum_{i=1}^m x_{ij} - \xi_j)\varphi_{D_j}(\xi_j)d\xi_j$$

is replaced with

$$\frac{h_j}{S}\sum_{s=1}^{S}(\sum_{i=1}^{m} x_{ij} - \xi_j^s)^+$$

Note that the expression $f(\bullet)^+$ means $\max(0, f(\bullet))$. Thus if the total of shipment, $(\sum_{i=1}^{m} x_{ij})$ is greater than the generated demand scenario (ξ_j^s) then there occurs an excess inventory. In another way around where the generated demand scenario is greater then the total shipment, the term $(\sum_{i=1}^{m} x_{ij} - \xi_j^s)^+$ would be zero. Thus $\frac{1}{S}\sum_{s=1}^{S}(\sum_{i=1}^{m} x_{ij} - \xi_j^s)^+$ means an average holding inventory over S scenarios generated. This replacing term is more attractive than the integration one. In the same manner, the inventory shortage cost term

$$p_j \int_{\sum_{i=1}^m x_{ij}}^{\infty} (\xi_j - \sum_{i=1}^m x_{ij})\varphi_{D_j}(\xi_j)d\xi_j$$

is replaced with

$$\frac{p_j}{S}\sum_{s=1}^{S}(\xi_j^s - \sum_{i=1}^{m} x_{ij})^+.$$

Then the objective function becomes

$$min \sum_{j=1}^{n}\sum_{s=1}^{S}\frac{h_j}{S}(\sum_{i=1}^{m} x_{ij} - \xi_j^s)^+ + \sum_{j=1}^{n}\sum_{s=1}^{S}\frac{p_j}{S}(\xi_j^s - \sum_{i=1}^{m} x_{ij})^+ + \sum_{j=1}^{n}\sum_{i=1}^{m} T_{ij}(x_{ij}) \qquad (4.1)$$

There are two complex terms, $(\sum_{i=1}^{m} x_{ij} - \xi_j^s)^+$ and $(\xi_j^s - \sum_{i=1}^{m} x_{ij})^+$, in the expression above. To simplify, we reformulate the problem by introducing two additional parameters, Z_j^{s+} and Z_j^{s-}, and three more constraints:

$$\begin{aligned} Z_j^{s+} &\geq \sum_{i \in I} x_{ij} - \xi_j^s & \forall j, s \quad (4.2)\\ Z_j^{s-} &\geq \xi_j^s - \sum_{i \in I} x_{ij} & \forall j, s \quad (4.3)\\ Z_j^{s+}, Z_j^{s-} &\geq 0 & \forall j, s \quad (4.4) \end{aligned}$$

The terms $(\sum_{i=1}^{m} x_{ij} - \xi_j^s)^+$ and $(\xi_j^s - \sum_{i=1}^{m} x_{ij})^+$ in problem (4.1) are replaced with Z_j^{s+} and Z_j^{s-}. These new two variables represent the holding inventory and inventory shortage amount respectively which are determined by the relation in constraints (4.2) and (4.3). The constraint (4.4) is added to indicate that Z_j^{s+} and Z_j^{s-} always have positive value or zero. It is obviously to see that Z_j^{s+} and Z_j^{s-} cannot be positive at the same time. For example, if the total shipment, $\sum_{i \in I} x_{ij}$, is greater than the scenario demand, ξ_j^s, then Z_j^{s+} will have a positive value and Z_j^{s-} is forced to be zero by constraint (4.4) through the minimizing process in problem (4.1).

Now the original problem can be expressed as follows:

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} \sum_{s=1}^{S} \frac{h_j}{S} Z_j^{s+} + \sum_{j=1}^{n} \sum_{s=1}^{S} \frac{p_j}{S} Z_j^{s-} + \sum_{j=1}^{n} \sum_{i=1}^{m} T_{ij}(x_{ij}) \\ s.t. \quad & \sum_{j=1}^{n} x_{ij} \leq C_i & \forall i \\ & \sum_{i=1}^{m} x_{ij} \leq Z_j^{s+} + \xi_j^s & \forall j, s \\ & \sum_{i=1}^{m} x_{ij} \geq \xi_j^s - Z_j^{s-} & \forall j, s \\ & x_{ij}, Z_j^{s+}, Z_j^{s-} \geq 0 & \forall i, j, s \end{aligned}$$

It should be noted that the inventory cost component is now a linear function.

4.2.2 Fixed Charge Transportation Cost

In this section, we will study a special case of the concave transportation cost. Assume that T_{ij} is the fixed charge transportation cost function and it can be shown as:

$$T_{ij}(x_{ij}) = \begin{cases} 0 & \text{if } x_{ij} = 0 \\ F_{ij} + v_{ij}x_{ij} & \text{if } x_{ij} > 0 \end{cases}$$

where F_{ij} is a fixed cost incurring whenever there is a flow from W_i to R_j. The v_{ij}, is a unit variable cost of transporting commodity from W_i to R_j.

We can transform this fixed charge transportation problem as a zero-one mixed integer problem (MIP) by introducing binary variables y_{ij}. The transportation cost T_{ij} has a form:

$$T_{ij}(x_{ij}) = F_{ij}y_{ij} + v_{ij}x_{ij}$$

with

$$y_{ij} = \begin{cases} 0 & \text{if } x_{ij} = 0 \\ 1 & \text{if } x_{ij} > 0 \end{cases}$$

Then our problem becomes:

$$\begin{aligned} \min \quad & \sum_{j=1}^{n}\left(\frac{h_j}{S}\sum_{s=1}^{S} Z_j^{s+} + \frac{p_j}{S}\sum_{s=1}^{S} Z_j^{s-}\right) + \sum_{j=1}^{n}\sum_{i=1}^{m}(F_{ij}y_{ij} + v_{ij}x_{ij}) \\ s.t. \quad & \sum_{j=1}^{n} x_{ij} \ \le \ C_i \qquad \forall i \\ & \sum_{i=1}^{m} x_{ij} \ \le \ Z_j^{s+} + \xi_j^s \qquad \forall j, s \end{aligned}$$

$$
\begin{aligned}
\sum_{i \in I} x_{ij} &\geq \xi_j^s - Z_j^{s-} && \forall j, s \\
x_{ij} &\leq C_i y_{ij} && \forall i, j \\
x_{ij}, Z_j^{s+}, Z_j^{s-} &\geq 0 && \forall i, j, s \\
y_{ij} &= \{0, 1\} && \forall i, j
\end{aligned}
$$

4.3 Modified Dynamic Slope Scaling Procedure

The Dynamic Slope Scaling Procedure (DSSP) was developed by Kim and Pardalos [34, 35]. This heuristic is an efficient and effective solution approach for the fixed charge and concave piecewise network flow problems. It can estimate the optimal solution with a small distance from the optimal solution in reasonable time. The DSSP approach does not attempt to solve the problem as a mixed integer problem but rather use a sequence of linear problems to estimate the optimal solution. The fixed charge cost is represented as a "linear" cost factor, $\bar{v}_{ij}$, where its relation is expressed as follows.

$$\bar{v}_{ij} = v_{ij} + \frac{F_{ij}}{\bar{x}_{ij}}$$

This linear cost factor reflects the slope of the cost function of linear problem. From the relation above, $\bar{v}_{ij}$ changes as $\bar{x}_{ij}$ changes. The procedure continues updating $\bar{v}_{ij}$ until $\bar{x}_{ij}$ can make the linear factor effectively represents the combination of variable cost v_{ij} and marginal fixed cost F_{ij} simultaneously. For example, at point $\bar{x}_{ij}$ in Figure 4.2, it shows that the solution of linear problem is equal to the solution of the fixed charge problem where the heuristic terminates. The computational test results reported in [34, 35] on the problem size up to 37 nodes 335 arcs show that the DSSP is efficient and reliable. The worst percentage difference between the best DSSP solution and the optimal solution is

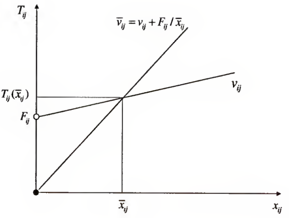


Figure 4.2: Dynamic slope scaling factor $\bar{v}_{ij}$

0.65%. Moreover, the DSSP finds the solution in about 4000 times faster than (measured by CPU time) an exact algorithm. The result also shows that the DSSP can solve the problem up to 202 nodes 10200 arcs in a little over one minute of CPU time.

The following DSSP steps are modified to test on our approximated problem.

Step 0: Initialize the "linear factor", $\bar{v}^0_{ij}$, to obtain an initial solution, $\bar{x}^0_{ij}$.

$$\bar{v}^0_{ij} = v_{ij} \quad \forall i, j$$

Then, solve for the initial solution using an LP.

Step 1: Update the "linear factor", $\bar{v}^{k+1}_{ij}$, at iteration *k+1*

$$\bar{v}^{k+1}_{ij} = \begin{cases} v_{ij} + \frac{F_{ij}}{\bar{x}^k_{ij}}, & \text{if } \bar{x}^k_{ij} > 0 \quad \text{at } k > 0 \\ v_{ij}, & \text{if } \bar{x}^k_{ij} = 0 \quad \text{at } k = 0 \\ \bar{v}^r_{ij}, & \text{if } \bar{x}^k_{ij} = 0 \quad \text{at } k \geq 1 \end{cases}$$

where r is the index of the most recent value of the slope scaling factor when $\bar{x}_{ij}r > 0$.

Step 2: Solve the problem using an LP with an updated cost vector.

Step 3: Observe the solution, $\bar{x}_{ij}^{k+1}$. If $\bar{x}_{ij}^{k+1} = \bar{x}_{ij}^{k}$ (indicate no further improvement) then terminates. Otherwise go to step 1.

4.4 Computational Experiments

This section describes how we measure the effectiveness of the DSSP heuristic to estimate the optimal solution for the approximated problem. The DSSP solution is compared with the optimal solution using Branch and Bound *(B & B)* method. Note that we transform the approximated problem to the mixed integer programming problem (MIP) before implementing the *B & B* algorithm. The two methods, DSSP and *B&B*, are coded in C with CPLEX 7.0 (callable library). The computational experiment is performed on an IBM SP PowerPC Platform two-processor node which is 332 MHz with 512 Mb of memory. The computational times spent on both methods are compared and reported. Finally, we can further evaluate the true objective value from DSSP solution in the original problem. This true objective value is the upper bound for the original problem.

4.4.1 Generating a Test Problem

Parameters are set up as follows:

- *Demand*

  Demands at each retailer are stochastic and possess the same distribution function. In this study, we do experiments on two distribution functions: uniform and

exponential. To generate demand scenarios, first we randomly generate the mean of demand at each retailer from the range (20, 50).

$$mean = \text{Uniform}(20, 50)$$

Then we generate a random number, *rn1*, from Uniform(0,1). This random number is used in the algorithm *IT-1* (Fig. 4.1) to obtain a scenario demand point. Let the cumulative distribution of demand (CDF) at retailer, R_j, be denoted by $\Phi(\xi_j) = \frac{\xi_j}{b}$, where b is an upper bound of the demand. Following the algorithm *IT-1*, the scenario demand ξ_j^s is obtained from

$$\xi_j^s = \text{rn1} * b$$

For exponential distribution, let the CDF be denoted by $\Phi(\xi_j) = 1 - e^{-\xi_j/mean}$. We obtain the randomly generated scenario demand points, ξ_j^s, as follows.

$$\xi_j^s = -ln(1 - \text{rn1}) * mean$$

.

- *Fixed Cost*

The fixed cost occurs whenever there is a shipment from a warehouse, W_i, to a retailer, R_j. The fixed cost for each link of warehouse and retailer, F_{ij}, is randomly created from the range (10, 30).

$$F_{ij} = \text{Uniform}(10, 30)$$

- *Variable Cost*

The variable cost is a cost per unit of shipping commodities from W_i to R_j. This

cost occurs with the fixed cost. In practice, the variable transportation cost mostly depends on the distance between the source and the destination. Thus we generate the variable cost based on the distance between warehouses and retailers. First we randomly assign a coordinate of every warehouse and retailer. Then the distance between every pair of W_i and R_j is calculated. Finally, the variable cost, v_{ij}, is given from a range (1,5) based on the calculated distance above.

- *Penalty and Holding Cost*

  The penalty and holding cost appear in the inventory cost component. In this study, the penalty and holding cost are 30 and 15 $ per unit respectively.

- Capacity at the W_i

  The only set of constraints in our problem is limited capacity at all warehouses. In the computational experiments, the capacity level, C_i, are the same at every warehouse. The capacity level is determined directly to the demand needed at all retailers but inversely to number of warehouses in each particular instance. For example, in 4 warehouses 5 retailers problem, the capacity at the warehouse tends to be more in high demand instances than the low demand instances. In addition, if the number of retailers is fixed, the capacity at each warehouse tends to be more in the problem that has less warehouses. We set the total capacity in each instance at 70 % of the demand range. We do not want to have too high capacity because the problem can become an uncapacitated problem, which is not in our interest. For the uniformly distributed demands problem, the capacity at each warehouse

is calculated from the equation below.

$$C_i = \left(\sum_{j=1}^{n} \bar{D}_j + 0.2 * n * (\text{half width of demand}) \right) / m$$

where $\bar{D}_j$ is the mean of demand at R_j. The capacity level for the exponentially distributed demands problem can be calculated from the equation below.

$$C_i = (ln(0.3) * (-\sum_{j=1}^{n} \bar{D}_j))/m$$

4.4.2 Computational Test Results

First we construct the test problems in three sets of various warehouse-retailer combinations. The three sets are two warehouses with various number of retailers, four warehouses with various number of retailers, and six warehouses with various number of retailers. For each combination in every set, we randomly generate 10 problem instances for 20, 40, 60, 80, and 100 demand scenarios respectively. Thus the DSSP heuristic is tested on 50 problem instances for every warehouse-retailer combination.

The computational results are reported in Table 4.1 - 4.6. The first three columns, *W, R, S*, indicate the number of *warehouses, retailers*, and *scenarios* respectively. Then the following columns, we report the average, minimum, and maximum value of relative errors (measuring the quality of the solution), and CPU times in seconds using *DSSP* heuristic and *B&B* algorithm respectively. The relative errors are calculated from

$$\%RE = \left(\frac{f_{DSSP} - f_{B\&B}}{f_{B\&B}} \right) * 100.$$

where f_{DSSP} is an objective value obtained from DSSP heuristic, and $f_{B\&B}$ is an optimal objective value obtained from *B&B*.

The results show that DSSP finds almost optimal solutions within very small amount of CPU times. For uniform demand case (see Table4.1, 4.3, and 4.5), the overall of the relative errors average for all test problems is about 1.56%, and the worst relative error reported is 8.72%. For exponential demand case (see Table 4.2, 4.4, and 4.6),DSSP gives better estimation. The overall average of relative errors is a little over 1%, and the worst relative error reported is 4.51%.

Figure 4.3 to 4.8 show the CPU times that are correlated with number of scenarios, number of retailers, and the number of warehouses. The *B&B* time increases more rapidly than the DSSP time. As a result, the gap between these two times is increasing exponentially with those two factors. It suggests that we can trade the solution quality with the computational time by obtaining a *good* solution from the DSSP within reasonable time instead of obtaining an optimal solution from *B&B* within longer period of computational time. It is also noted that *B&B* times in exponential demand case are much less than the uniform one (especially in problems of 4 or more warehouses), while DSSP times are almost the same in most cases for both demand distributions.

Table 4.1: Summary of computational results for two warehouses with various number of retailers problems with uniformly distributed demands

Size						CPU Time (seconds)					
		S	%RE			DSSP			*B & B*		
W	R		**AVG**	Min	Max	**AVG**	Min	Max	**AVG**	Min	Max
2	5	20	**1.1557**	0.0080	4.1186	**0.022**	0.020	0.030	**0.106**	0.030	0.200
		40	**0.9628**	0.0282	4.0056	**0.039**	0.030	0.050	**0.199**	0.060	0.320
		60	**0.5131**	0.0630	1.4973	**0.071**	0.060	0.080	**0.433**	0.120	0.770
		80	**0.8882**	0.0713	2.2603	**0.109**	0.080	0.130	**0.687**	0.150	1.370
		100	**0.8716**	0.0181	3.6340	**0.149**	0.120	0.180	**0.734**	0.210	1.160
2	10	20	**0.5304**	0.0742	1.1672	**0.043**	0.040	0.060	**0.188**	0.070	0.300
		40	**0.4537**	0.1000	1.8622	**0.099**	0.090	0.110	**0.305**	0.140	0.760
		60	**0.7006**	0.0854	1.9774	**0.179**	0.140	0.220	**0.574**	0.240	1.280
		80	**0.6656**	0.1276	1.8799	**0.274**	0.210	0.310	**1.090**	0.400	2.240
		100	**0.8570**	0.1363	2.2631	**0.410**	0.320	0.540	**1.468**	0.550	3.010
2	15	20	**0.2967**	0.0754	0.8716	**0.073**	0.060	0.090	**0.293**	0.120	0.940
		40	**0.3537**	0.0637	0.8047	**0.191**	0.170	0.240	**0.893**	0.240	2.330
		60	**0.4209**	0.1215	0.8570	**0.310**	0.260	0.390	**1.126**	0.440	2.620
		80	**0.3434**	0.0632	1.1555	**0.540**	0.400	0.670	**1.777**	0.660	3.300
		100	**0.4024**	0.1091	1.1142	**0.785**	0.560	0.910	**2.535**	0.960	4.680
2	20	20	**0.4277**	0.0154	0.7780	**0.103**	0.090	0.130	**0.330**	0.170	0.840
		40	**0.2960**	0.0955	0.8515	**0.241**	0.190	0.290	**0.945**	0.350	1.910
		60	**0.3924**	0.0789	1.2349	**0.498**	0.400	0.590	**1.991**	0.820	3.860
		80	**0.3957**	0.1100	0.9331	**0.729**	0.540	0.990	**2.589**	1.400	3.710
		100	**0.3883**	0.1143	0.8033	**1.256**	1.120	1.410	**4.321**	1.400	7.760

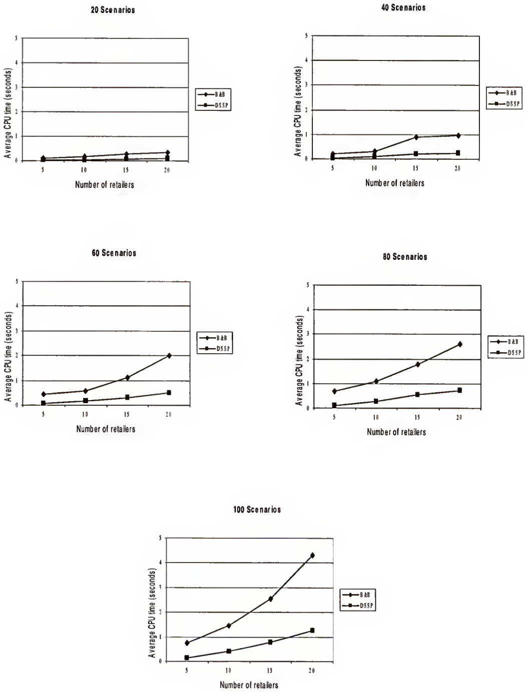


Figure 4.3: Average CPU times for two warehouses with uniformly distributed demands

Table 4.2: Summary of computational results for two warehouses with various number of retailers problems with exponentially distributed demands

						CPU Time (seconds)					
Size			%RE			DSSP			*B & B*		
W	R	S	**AVG**	Min	Max	**AVG**	Min	Max	**AVG**	Min	Max
2	5	20	**0.5255**	0.0443	1.8792	**0.020**	0.010	0.030	**0.070**	0.030	0.180
		40	**0.4326**	0.1055	1.8971	**0.044**	0.040	0.050	**0.134**	0.060	0.350
		60	**0.4366**	0.0524	1.9914	**0.084**	0.070	0.100	**0.204**	0.110	0.430
		80	**0.5557**	0.0595	1.6385	**0.120**	0.110	0.130	**0.391**	0.190	0.960
		100	**0.3862**	0.0708	1.2470	**0.162**	0.120	0.200	**0.438**	0.240	0.700
2	10	20	**0.6868**	0.0899	2.0641	**0.043**	0.040	0.050	**0.140**	0.060	0.260
		40	**0.4566**	0.1017	1.7330	**0.107**	0.090	0.130	**0.273**	0.160	0.680
		60	**0.3180**	0.1323	1.0978	**0.191**	0.170	0.210	**0.507**	0.270	1.060
		80	**0.2429**	0.1653	0.3490	**0.278**	0.210	0.320	**0.716**	0.410	1.230
		100	**0.3604**	0.1444	0.7553	**0.427**	0.310	0.520	**1.114**	0.520	2.100
2	15	20	**0.3311**	0.0870	0.8152	**0.074**	0.060	0.090	**0.200**	0.120	0.510
		40	**0.4700**	0.1775	1.4629	**0.185**	0.150	0.240	**0.614**	0.240	1.240
		60	**0.2761**	0.1370	0.6244	**0.297**	0.260	0.330	**0.969**	0.450	2.120
		80	**0.3557**	0.1477	0.8104	**0.529**	0.420	0.600	**1.540**	0.750	2.870
		100	**0.4306**	0.1413	0.7505	**0.831**	0.570	0.960	**2.579**	1.030	4.870
2	20	20	**0.2927**	0.0676	0.6710	**0.102**	0.090	0.130	**0.291**	0.160	0.610
		40	**0.3481**	0.1267	0.6810	**0.251**	0.200	0.300	**0.806**	0.380	1.540
		60	**0.3022**	0.1471	0.6269	**0.520**	0.400	0.570	**1.977**	0.790	3.410
		80	**0.3197**	0.1505	0.6800	**0.759**	0.560	0.970	**2.980**	1.230	5.320
		100	**0.2424**	0.0435	0.4989	**1.158**	1.030	1.290	**3.766**	1.420	6.320

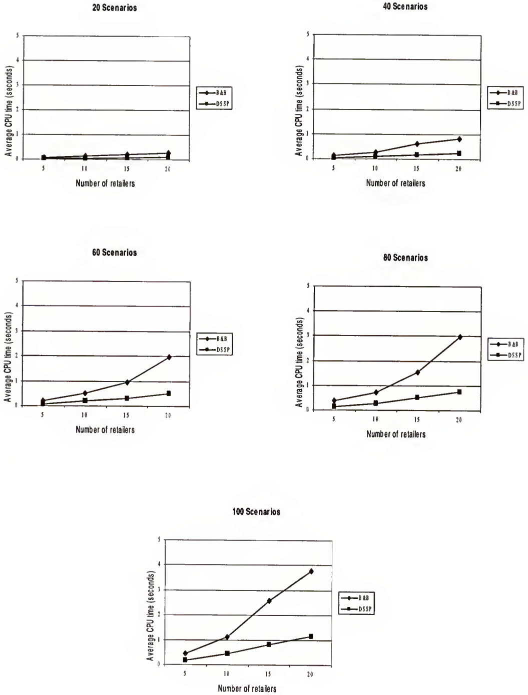


Figure 4.4: Average CPU times for two warehouses with exponentially distributed demands

Table 4.3: Summary of computational results for four warehouses with various number of retailers problems with uniformly distributed demands

Size			%RE			CPU Time (seconds)					
						DSSP			*B & B*		
W	R	S	**AVG**	Min	Max	**AVG**	Min	Max	**AVG**	Min	Max
4	5	20	**3.1550**	0.0624	7.0448	**0.027**	0.020	0.040	**0.397**	0.040	0.670
		40	**2.0444**	0.8306	4.1102	**0.054**	0.040	0.070	**0.995**	0.100	2.380
		60	**3.4927**	0.0392	8.7153	**0.093**	0.070	0.110	**1.341**	0.120	2.110
		80	**2.8230**	1.6397	5.6446	**0.147**	0.120	0.190	**2.269**	0.210	4.900
		100	**2.0033**	0.0349	6.0951	**0.192**	0.180	0.210	**3.688**	0.270	7.340
4	10	20	**1.7491**	0.0893	3.7347	**0.059**	0.050	0.080	**0.559**	0.110	1.370
		40	**1.8419**	0.6801	3.0623	**0.140**	0.110	0.170	**1.937**	0.450	4.040
		60	**2.1294**	1.1614	4.1193	**0.245**	0.180	0.330	**2.837**	0.690	8.260
		80	**1.6521**	0.3801	3.4317	**0.378**	0.290	0.430	**4.439**	1.020	9.450
		100	**1.5688**	0.1414	2.4974	**0.528**	0.440	0.580	**5.875**	2.590	9.810
4	15	20	**1.4958**	0.3943	2.7818	**0.102**	0.090	0.120	**1.142**	0.410	2.810
		40	**1.3508**	0.4821	3.0954	**0.231**	0.190	0.260	**2.498**	0.790	4.640
		60	**1.1489**	0.1008	2.0451	**0.427**	0.290	0.530	**4.841**	0.790	10.770
		80	**1.2326**	0.1816	1.8701	**0.692**	0.550	0.840	**5.660**	1.530	11.550
		100	**1.4405**	0.5844	2.8248	**1.000**	0.890	1.140	**10.472**	3.730	34.250
4	20	20	**1.4637**	0.4366	3.3957	**0.166**	0.12	0.28	**1.420**	0.450	2.880
		40	**1.1566**	0.5131	3.4098	**0.324**	0.27	0.39	**3.175**	1.260	6.240
		60	**1.3428**	0.7451	2.3536	**0.696**	0.45	0.9	**5.075**	1.910	8.650
		80	**1.4624**	0.4393	2.7556	**1.03**	0.81	1.26	**7.422**	3.120	12.320
		100	**1.0228**	0.4141	2.0100	**1.389**	0.94	1.65	**12.591**	5.000	21.580

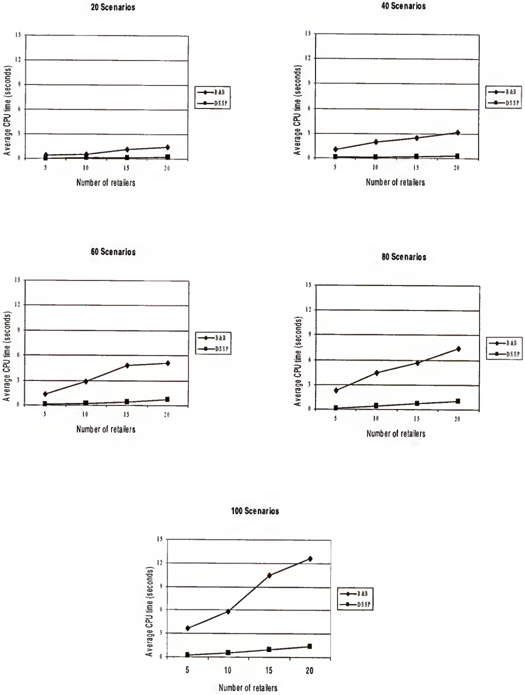


Figure 4.5: Average CPU times for four warehouses with uniformly distributed demands

Table 4.4: Summary of computational results for four warehouses with various number of retailers problems with exponentially distributed demands

Size			%RE			CPU Time (seconds)					
						DSSP			*B & B*		
W	R	S	**AVG**	Min	Max	**AVG**	Min	Max	**AVG**	Min	Max
4	5	20	**1.7428**	0.0177	3.3177	**0.030**	0.020	0.040	**0.256**	0.050	0.420
		40	**1.3808**	0.0035	3.7240	**0.057**	0.050	0.070	**0.731**	0.090	1.790
		60	**1.2829**	0.0353	4.5079	**0.105**	0.090	0.120	**1.320**	0.140	3.640
		80	**0.9755**	0.0040	2.8650	**0.146**	0.130	0.170	**1.968**	0.210	4.890
		100	**0.8628**	0.0854	1.7553	**0.198**	0.160	0.240	**2.630**	0.390	7.030
4	10	20	**0.9616**	0.2780	2.8569	**0.063**	0.050	0.080	**0.389**	0.090	1.020
		40	**0.5753**	0.1496	1.1220	**0.132**	0.100	0.160	**1.091**	0.210	3.050
		60	**1.0130**	0.1347	2.4087	**0.236**	0.210	0.280	**1.620**	0.340	3.550
		80	**1.3246**	0.3558	2.5186	**0.358**	0.270	0.430	**3.006**	0.550	6.400
		100	**1.4340**	0.0535	2.5802	**0.507**	0.390	0.630	**2.947**	0.850	5.240
4	15	20	**0.7604**	0.4027	1.4175	**0.101**	0.080	0.120	**0.586**	0.150	1.010
		40	**0.9554**	0.2179	1.6793	**0.228**	0.210	0.250	**1.251**	0.400	1.920
		60	**0.9915**	0.5711	2.0682	**0.392**	0.310	0.460	**2.690**	1.540	5.810
		80	**0.7735**	0.1273	2.5938	**0.630**	0.500	0.780	**3.431**	0.960	8.680
		100	**0.8063**	0.2480	1.2269	**0.938**	0.630	1.210	**5.905**	1.990	13.380
4	20	20	**0.7629**	0.2905	1.4914	**0.148**	0.130	0.170	**0.927**	0.420	1.400
		40	**0.6179**	0.3221	1.2095	**0.303**	0.270	0.410	**1.837**	0.780	2.420
		60	**0.6011**	0.2487	1.1601	**0.614**	0.500	0.720	**3.998**	0.980	7.480
		80	**0.5761**	0.2051	0.9224	**0.894**	0.690	1.090	**5.199**	3.010	10.880
		100	**0.7463**	0.1092	1.3717	**1.427**	0.990	1.690	**8.381**	4.980	11.050

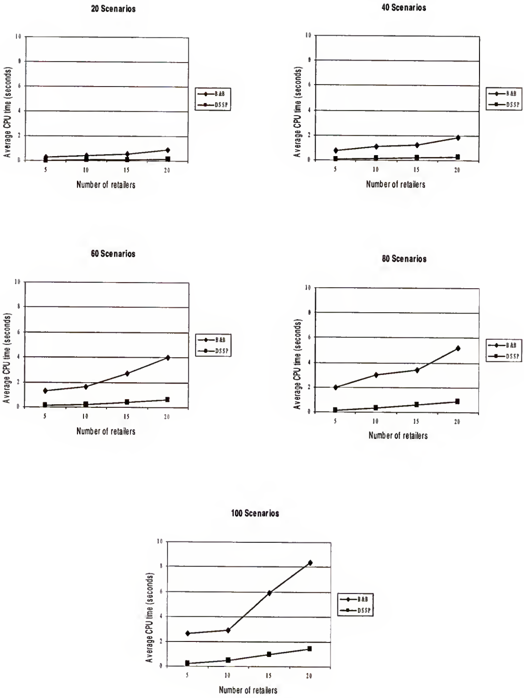


Figure 4.6: Average CPU times for four warehouses with exponentially distributed demands

Table 4.5: Summary of computational results for six warehouses with various number of retailers problems with uniformly distributed demands

Size			%RE			CPU Time (seconds) DSSP			CPU Time (seconds) *B & B*		
W	R	S	AVG	Min	Max	AVG	Min	Max	AVG	Min	Max
6	10	20	**2.8736**	0.4948	6.3081	**0.087**	0.080	0.100	**2.689**	0.900	8.730
		40	**3.0467**	0.1279	5.2134	**0.192**	0.160	0.230	**3.882**	1.280	12.310
		60	**2.7557**	1.3183	6.0911	**0.342**	0.320	0.370	**8.535**	2.830	22.670
		80	**2.9162**	0.9699	5.1529	**0.504**	0.420	0.570	**11.934**	3.630	39.830
		100	**2.1722**	0.8534	4.2993	**0.745**	0.630	0.920	**16.917**	7.540	47.150
6	15	20	**2.4060**	0.3136	5.7368	**0.163**	0.12	0.24	**4.327**	0.600	8.210
		40	**2.2218**	0.6727	3.7334	**0.353**	0.24	0.46	**20.235**	1.750	76.660
		60	**1.8268**	0.1750	3.2026	**0.618**	0.46	0.82	**24.954**	6.110	61.100
		80	**1.9758**	0.7237	3.0302	**1.022**	0.72	1.18	**38.161**	4.180	99.480
		100	**1.8670**	0.4709	3.2891	**1.462**	1.24	1.85	**61.666**	11.760	296.970
6	20	20	**1.9504**	1.1308	3.0116	**0.216**	0.180	0.280	**7.053**	1.290	14.960
		40	**2.6048**	1.5530	4.0621	**0.500**	0.380	0.740	**8.675**	2.850	25.970
		60	**1.8850**	0.1864	4.1537	**0.942**	0.830	1.100	**24.364**	3.810	96.340
		80	**2.0153**	0.6586	2.9714	**1.608**	1.380	2.020	**35.497**	6.000	85.390
		100	**1.8608**	0.6423	4.1247	**2.123**	1.57	2.75	**58.539**	10.910	296.310
6	25	20	**1.2644**	0.2980	2.1476	**0.272**	0.25	0.33	**6.683**	1.150	17.300
		40	**1.9190**	0.7186	4.1319	**0.714**	0.57	0.87	**14.630**	4.020	46.770
		60	**1.5461**	0.6172	3.2669	**1.294**	1.04	1.55	**43.851**	5.760	130.800
		80	**1.3478**	0.4458	2.1347	**1.968**	1.66	2.18	**39.871**	11.260	126.710
		100	**1.5610**	0.7446	2.5905	**3.77**	2.86	4.96	**78.839**	18.390	238.990

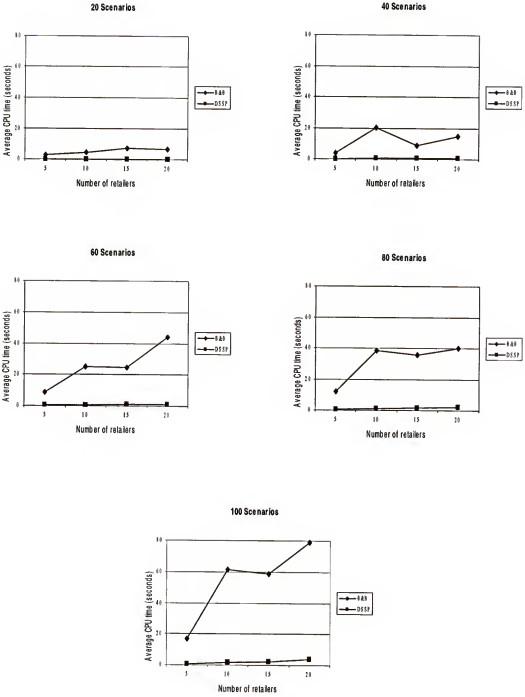


Figure 4.7: Average CPU times for six warehouses with uniformly distributed demands

Table 4.6: Summary of computational results for six warehouses with various number of retailers problems with exponentially distributed demands

Size			%RE			CPU Time (seconds)					
						DSSP			*B & B*		
W	R	S	**AVG**	Min	Max	**AVG**	Min	Max	**AVG**	Min	Max
6	10	20	**2.5388**	1.9029	3.1522	**0.090**	0.070	0.110	**1.112**	0.420	3.440
		40	**1.9064**	0.1773	2.7867	**0.190**	0.150	0.210	**1.972**	0.620	3.500
		60	**1.8935**	0.4207	3.5300	**0.321**	0.290	0.340	**3.825**	1.690	9.570
		80	**1.6101**	0.4933	3.9316	**0.501**	0.370	0.640	**5.413**	1.640	12.880
		100	**1.7727**	0.1483	2.9416	**0.730**	0.480	0.930	**8.556**	1.860	19.160
6	15	20	**1.4586**	0.5904	2.4502	**0.158**	0.120	0.190	**1.933**	0.600	3.440
		40	**0.8574**	0.1996	1.7724	**0.325**	0.250	0.390	**3.990**	1.200	7.810
		60	**1.1786**	0.5380	1.9634	**0.552**	0.420	0.710	**7.183**	3.230	16.530
		80	**1.1247**	0.4988	1.7884	**0.961**	0.800	1.110	**8.798**	4.230	24.540
		100	**1.2921**	0.7280	1.7849	**1.340**	1.160	1.560	**12.407**	5.290	29.530
6	20	20	**1.0230**	0.5958	1.6948	**0.196**	0.170	0.240	**2.316**	1.000	5.860
		40	**1.1973**	0.3326	1.7426	**0.478**	0.390	0.570	**6.315**	1.830	10.600
		60	**0.9412**	0.3608	1.7230	**0.906**	0.760	1.060	**6.593**	3.150	16.590
		80	**0.9563**	0.5089	1.7028	**1.417**	1.170	1.820	**14.139**	9.060	26.220
		100	**1.1525**	0.5535	1.6547	**1.847**	1.490	2.120	**19.827**	10.360	41.750
6	25	20	**0.8892**	0.3865	1.3793	**0.285**	0.220	0.360	**2.966**	1.210	5.340
		40	**0.8931**	0.5065	1.4050	**0.641**	0.540	0.770	**5.369**	2.820	8.420
		60	**0.7400**	0.3442	1.1686	**1.077**	0.920	1.300	**10.838**	4.970	21.110
		80	**0.7684**	0.3437	1.0914	**1.752**	1.240	2.430	**14.614**	9.040	22.920
		100	**0.7934**	0.4101	1.1201	**3.467**	2.790	4.360	**21.540**	13.950	38.130

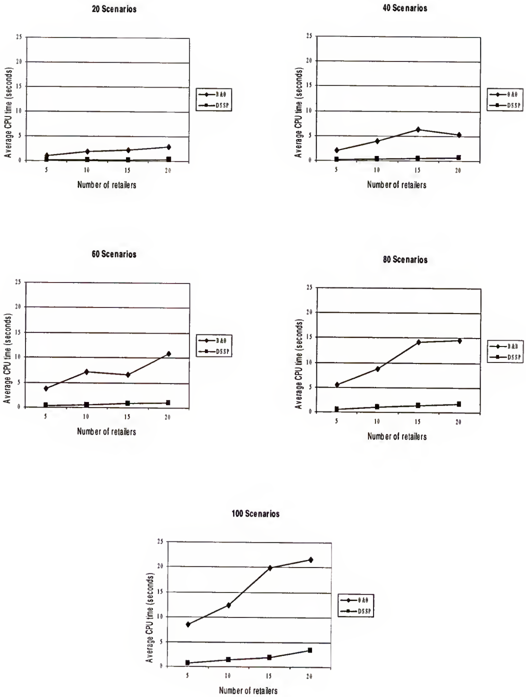


Figure 4.8: Average CPU times for six warehouses with exponentially distributed demands

4.4.3 Evaluating the True Objective Value

Our main interest is the true objective value. The feasible set of solution in the approximated problem is the same set in the original problem since they are bounded by the same set of constraints. Thus the DSSP solution is also a feasible solution to the original problem. We can evaluate the true objective value by borrowing the DSSP solution and substituting it in the original objective function.

$$\sum_{j=1}^{n}\{(h_j+p_j)\int_0^{\sum_{i=1}^{m} x_{ij}}\Phi_{D_j}(\xi_j)d\xi_j + p_j\bar{D}_j - p_j(\sum_{i=1}^{m} x_{ij})\} + \sum_{j=1}^{n}\sum_{i=1}^{m}(v_{ij}x_{ij} + F_{ij}1_{x_{ij}>0}) \tag{4.5}$$

where $1_{x_{ij}>0}$ is an indicator function. Its value is one when $x_{ij} > 0$, otherwise zero. Note that this is another form of the objective function. It is more convenient to evaluate the objective value from this form. From the computation experiments, we have DSSP solution for uniformly and exponentially distributed demand. Let $\Phi_U(\xi_j) = \frac{\xi_j}{b}$ and $\Phi_{EX}(\xi_j) = 1 - e^{-\xi_j/\lambda}$ be the CDF of uniform and exponential distribution respectively. Substituting the demand distribution in Equation 4.5, the objective function becomes

$$\sum_{j=1}^{n}\{\frac{(h_j+p_j)}{b}(\frac{(\sum_{i=1}^{m} x_{ij})^2}{2}) + p_j\bar{D}_j - p_j(\sum_{i=1}^{m} x_{ij})\} + \sum_{j=1}^{n}\sum_{i=1}^{m}(v_{ij}x_{ij} + F_{ij}1_{x_{ij}>0})$$

for the uniform demand and

$$\sum_{j=1}^{n}\{(h_j+p_j)(\sum_{i=1}^{m} x_{ij} - \frac{e^{\lambda(\sum_{i=1}^{m} x_{ij})}}{\lambda} - \frac{1}{\lambda}) + p_j\bar{D}_j - p_j(\sum_{i=1}^{m} x_{ij})\} + \sum_{j=1}^{n}\sum_{i=1}^{m}(v_{ij}x_{ij} + F_{ij}1_{x_{ij}>0})$$

for the exponential demand.

4.5 Conclusions

This chapter approximates the integral components, inventory holding cost and penalty cost component, in the objective function using Monte Carlo method. The solution of this approximated problem with a fixed charge transportation cost is estimated by DSSP heuristic. The heuristic is tested with two types of demand distribution function: uniform and exponential. Experimenting with various combinations of warehouses and retailers, the results show that the heuristic finds solutions faster than the *B & B* method especially in large problems. To measure the quality of the heuristic solutions, we find a relative differences between the heuristic and optimal solution. It shows that the heuristic can solve the approximated problem optimally or near optimality. It is noted that the heuristic provides smaller relative differences in exponentially distributed demands than the uniform ones. It means that the heuristic works better with the exponentially distributed demands case. Once we obtain the heuristic solution of the approximated problem, the upper bounds of the original problem can be evaluated from this solution.

CHAPTER 5
THE LAGRANGIAN RELAXATION MODEL

5.1 Introduction

This chapter extends the study in the fixed charge transportation cost problem from the previous chapter. Thus far the only known fact about the problem is the upper bound while the optimal solution is still unknown. In this case the quality of upper bound obtained cannot be determined. As an alternative, the gap between the upper bound and lower bound can be used as an indication of the upper bound quality. Thus the goal of this chapter is to compare the upper bound obtained from the previous chapter with the lower bound obtained in this chapter. In this study, the lower bound is derived by Lagrangian relaxation technique, which is one of the most successful techniques in providing the lower bound. It is a general solving strategy [2] and easily applied to wide variety of integer programming problems. In addition, the Lagrangian relaxation provides much better bounds than the linear programming (LP) relaxation.

This chapter is organized as follows. Section 5.2 presents a literature review of Lagrangian relaxation in many diverse areas. Section 5.3 describes the general Lagrangian relaxation procedure and subgradient optimization. Section 5.4 provides the Lagrangian relaxation model for computing lower bounds. Section 5.5 reports computational results. Finally the conclusions are presented in Section 5.6.

5.2 Literature Review

The purpose of this section is not to attempt an exhaustive review, but rather to survey the broad variety of Lagrangian relaxation ideas and several possibilities of their applications.

The term “Lagrangian Relaxation” is coined by Geoffrion [27]. The key idea of Lagrangian relaxation is to remove complicating constraints from hard integer programming problems. As a result, the problems could “easily” be solved [24, 28, 9]. In many cases, the mathematical models can be decomposed to subproblems when the Lagrangian relaxation method is applied. Typically, the subproblem can be efficiently solved by any existing methodology or algorithm [2]. For example, the primary work in this area by Held and Karp [31] is done in the traveling-salesman problem where its Lagrangian problem is based on minimum spanning trees. In [24], it shows an evidence that the Lagrangian problem of the generalized assignment problem can be dealt with knapsack solution procedure.

Fisher [25] presents the generic Lagrangian relaxation algorithm which consists of three steps: (1) construction of branch and bound tree, (2) adjustment of Lagrange multipliers, and (3) solution of Lagrangian problem. The first step provides the best known feasible value which is passed to step 2 and 3. The last two steps are iterated until an iteration limit is reached or a better bound is discovered. Furthermore, he commented that the relaxed problem should be easier to solve but not too easy. It implies that the choice of relaxed constraints also plays an important role in the success of the relaxation. On top of that, there is a question of how to compute good multipliers. The answer to this question is that the most utilized and general method is the subgradient method.

Finally, since some constraints are eliminated from the problem formulation, the Lagrangian subproblem solution may not be feasible to the original problem. Practically, there is no general rule of thumb to recover the feasible solution from the Lagrangian solution. It is a problem-dependent procedure.

As reviewed in Fisher [24], literature oriented to the applications of Lagrangian relaxation during 1970s. The Lagrangian relaxation was derived to variety integer programming problems including the traveling salesman problem, the general integer problem, the location problem, the general assignment problem, etc. The computational experience showed that the Lagrangian relaxation provided extremely sharp bounds.

The Lagrangian relaxation has become a heavily used tool in combinatorial problems. In the last decade, more and more literature have devoted to prove the power of Lagrangian relaxation method in many applications. An application in the location problem is successfully done by Beasley [10]. His computational results indicate the robustness of Lagrangian relaxation-based heuristic in four different location problems: p-median, uncapacitated warehouse location, capacitated warehouse location and capacitated warehouse location with single source constraints.

Following Beasley, Tragantalernsak et al. [57] propose six Lagrangian-based heuristics for the two-echelon, single-source, capacitated facility location problem. The major difference in these six heuristics is the sets of relaxed constraints. The computational results have shown that the relaxed constraints affect the solution quality. This study is a significant example of deciding which constraints should be relaxed. In their second paper [58], they develop a branch and bound algorithm based on the most efficient Lagrangian heuristic. The numerical results show that the new algorithm requires smaller

branch and bound tree size and CPU time than a standard LP based zero-one integer programming package.

The application of Lagrangian relaxation in the lot-sizing problem is done by Afentakis et al. [1]. They consider this problem in multistage assembly systems. Their model is formulated in term of "echelon stocks." Applied the Lagrangian relaxation technique, the formulation is decomposed into single stage subproblems. The Lagrangian problem provides sharp bounds which are consequently used to develop an efficient Branch and Bound algorithm.

Balakrishnan and Graves [7] applied the Lagrangian relaxation to develop a composite algorithm in a multicommodity network flow problem in which the arc costs are piecewise linear concave function.

Recently, Balakrishnan and Geunes [6] propose a Lagrangian-based heuristic for the flexible demand assignment problem *(FDA)* which is formulated as a mixed-integer program. The heuristic generates the feasible solution from the decomposed Lagrangian subproblem, which can be solved as a knapsack problem with expandable items *(KPEI)*. Their computational results show that the upper bound obtained is tighter than the linear programing relaxation.

5.3 Lagrangian Relaxation

This section describes the Lagrangian relaxation procedure and the detail in the subgradient algorithm employed to obtain the best lower bound.

To demonstrate, consider the integer problem:

(IP)

$$\begin{aligned} Z_{IP} &= \min cx & \\ s.t. \quad & Ax & \leq b \\ & x & \in Z^+ \end{aligned}$$

The problem *(IP)* has one set of explicit linear inequality constraints. The decision variables x is a nonnegative integer.

Suppose that the problem *(IP)* is easier to solve if the explicit constraints are removed or relaxed. These relaxed constraints are brought into the objective function with associated penalty cost which is represented by the Lagrange multipliers, λ. Thus the *Lagrangian relaxation* or *Lagrangian subproblem* is created as follows.

(LR)

$$\begin{aligned} Z_{LR} &= \min cx + \lambda(Ax - b) \\ s.t. \quad & x \in Z^+ \end{aligned}$$

or we can refer it as the *Lagrangian function*

$$L(x, \lambda) = \min\{cx + \lambda(Ax - b) : x \in Z^+\}.$$

As mentioned before, the Lagrangian subproblem can be solved by exploiting available procedures or algorithms for a given set of Lagrange multipliers. By weak duality theorem [51], the optimal objective value obtained provides a lower bound to the original problem *(IP)*. It implies that $Z_{LR} \leq Z_{IP}$. Since the primal problem is a minimization problem, we wish to find the greatest lower bound. This can be achieved through solving

the *Lagrangian dual* or the *Lagrangian multiplier* problem [2]. This problem concerns with choosing the multipliers that provide the sharpest bound. It is defined as

(LD)

$$Z_{LD} = \max_{\lambda \geq 0} L(x, \lambda)$$

The problem LD is a max-min problem. We can use an iterative approach to obtain the best lower bound from this relaxation. That is, for a fixed value of λ we can solve $L(x, \lambda)$. Then to maximize Z_{LR}, we need to find an improving direction of λ and take a step in that direction. Then we resolve the subproblem again with an updated λ. The most common algorithm used is the *subgradient optimization*(see details in [31]). This algorithm ensures that λ converges to the value that maximize Z_{LR}.

5.3.1 Modified Subgradient Optimization Algorithm

In this subsection, we modify the algorithm before implementing to our problem. Let LBD and UBD be a lower and upper bound on the optimal value to the original problem respectively. UBD is any feasible solution. Let t and d_t denote an iteration counter and the step length parameter at iteration t respectively. The description of the algorithm is shown below.

Step 1:

$$\begin{cases} \text{Set} & LBD & = & -\infty \\ & t & = & 0 \\ & d_0 & = & 2 \\ \text{Initialize} & \lambda^0 & = & 0 \end{cases}$$

Note that *LBD* is determined by the best Lagrangian dual value and *UBD* is determined by the best feasible solution found so far. The step scale (d_t) is halved every fixed number of consecutively iterations where the *LBD* does not improve.

Step 2: Solve $\min_{x \in \mathbf{X}} L(\mathbf{x}, \lambda)$ and let $\mathbf{x}^*$ be the solution and set

$$LBD = \max\{LBD, L(\mathbf{x}^*, \lambda^{\mathbf{t}})\}$$

Step 3: Check for stopping criteria

If $\lambda^t - \lambda^{t-1} < \varepsilon$ for five consecutive iterations or $t = t_{max}$, stop. The current dual solution is chosen as the best *LBD*.

Step 4: Compute subgradient

$$\gamma_i = \sum_j x_{ij}^* - C_i \ \ \forall i$$

Step 5: Compute step size

$$T_t = d_t(UBD - L(\mathbf{x}, \lambda)) / \parallel \gamma \parallel^2$$

Step 6: Update the *Lagrange* multipliers

$$\lambda^{t+1} = \lambda^t + T_t \gamma$$

Step 7: Update iteration counter *(t)*

Note that unlike the generic subgradient algorithm, we do not attempt to find a feasible solution in the algorithm to update *UBD*. We expect to obtain the maximum lower bound from the subgradient method. As a reminder, the purpose of our study is to measure the upper bound quality by comparing with the best lower bound obtained from the Lagrangian problem.

5.4 Lagrangian Relaxation Model

Consider the capacitated inventory and transportation problem.

(P)

$$\min \sum_{j=1}^{n}\{(h_j + p_j) \int_0^{\sum_{i=1}^{m} x_{ij}} \Phi_{D_j}(\xi_j) d\xi_j + p_j \bar{D}_j - p_j(\sum_{i=1}^{m} x_{ij})\} + \sum_{j=1}^{n}\sum_{i=1}^{m} T_{ij}(x_{ij}) \quad (5.1)$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leq C_i \quad \forall i \quad (5.2)$$

$$x_{i,j} \geq 0 \quad \forall i, j \quad (5.3)$$

This problem *(P)* is difficult to solve computationally. However, the problem becomes an uncapacitated problem and could be "easily" solved if the set of capacity constraints (5.2) is eliminated. Let λ_i denote Lagrangian multipliers associated with the set of capacity constraints (5.2). Then, the Lagrangian relaxation problem (*LR*) can be expressed as follows.

(LR)

$$\begin{aligned} \min \quad & \sum_{j=1}^{n}\{(h_j + p_j) \int_0^{\sum_{i=1}^{m} x_{ij}} \Phi_{D_j}(\xi_j) d\xi_j + p_j \bar{D}_j - p_j(\sum_{i=1}^{m} x_{ij})\} + \\ & \sum_{j=1}^{n}\sum_{i=1}^{m} T_{ij}(x_{ij}) + \sum_{i=1}^{m} \lambda_i (\sum_{j=1}^{n} x_{ij} - C_i) \end{aligned}$$

or

$$\begin{aligned} \min \quad & \sum_{j=1}^{n}\{(h_j + p_j) \int_0^{\sum_{i=1}^{m} x_{ij}} \Phi_{D_j}(\xi_j) d\xi_j + p_j \bar{D}_j - \sum_{i=1}^{m} (p_j - \lambda_i) x_{ij} + \\ & \sum_{i=1}^{m} T_{ij}(x_{ij})\} - \sum_{i=1}^{m} \lambda_i C_i \end{aligned}$$

subject to

$$x_{i,j} \geq 0 \quad \forall i, j$$

The Lagrangian function is expressed as follows.

$$L(\mathbf{x}, \lambda) = \min \left\{ \sum_{j=1}^{n} \left((h_j + p_j) \int_0^{\sum_{i=1}^m x_{ij}} \Phi_{D_j}(\xi_j) d\xi_j + p_j \bar{D}_j - \sum_{i=1}^{m} (p_j - \lambda_i) x_{ij} + \sum_{i=1}^{m} T_{ij}(x_{ij}) \right) - \sum_{i=1}^{m} \lambda_i C_i : x_{i,j} \geq 0 \right\} \tag{5.4}$$

For given multipliers, we can solve the Lagrangian problem (5.4) optimally. The optimal solution of this Lagrangian problem provides us a lower bound to the original problem *(P)*. However, it does not guarantee that the chosen λ will give us a good lower bound. To maximize this lower bound, we can solve to dual problem of (5.4). Now, let $q(\lambda)$ be the dual function of $\min_{x \in \mathbf{X}} L(\mathbf{x}, \lambda)$. Then the dual problem corresponding to the problem (5.4) is

(LD)

$$\max q(\lambda) \tag{5.5}$$

$$st. \quad \lambda \geq 0.$$

As discussed eariler, the subgradient method is employed to obtain the best lower bound.

It is interesting that for given λ_i for $i = 1, \ldots, m$, it is noted that the term $\sum_{i=1}^{m} \lambda_i C_i$ in function $L(\mathbf{x}, \lambda)$ is a constant at this point. Then the problem LR is decomposed into n uncapacitated single-retailer problems (where we have dropped the retailer index j).

The decomposed function can be expressed as follows.

$$\min_{x_i \geq 0} \left((h+p) \int_0^{\sum_{i=1}^m x_i} \Phi_D(\xi) d\xi + p\bar{D} - \sum_{i=1}^m (p-\lambda_i) x_i + \sum_{i=1}^m T_i(x_i) \right) \tag{5.6}$$

5.4.1 Fixed Charge Transportation cost

Now let us incorporate the fixed charge transportation cost, $T_i(x_i)$, to our model. $T_i(x_i)$ is defined as follows.

$$T_i(x_i) = \begin{cases} 0 & \text{if } x_i = 0 \\ F_i + v_i x_i & \text{if } x_i > 0 \end{cases}$$

Substitute $T_i(x_i)$ in 5.6, we obtain

$$\min_{x_i \geq 0} \left((h+p) \int_0^{\sum_{i=1}^m x_i} \Phi_D(\xi) d\xi + p\bar{D} - \sum_{i=1}^m (p-\lambda_i) x_i + \sum_{i=1}^m (F_i + v_i x_i) \right) \tag{5.7}$$

Since we encounter with a non-differentiable function, the minimum cannot be found by directly differentiating. Solving the subproblem above is not trivial. However, we can exploit the theorem below to obtain an optimal solution to the Lagrangian subproblem.

Theorem 1 *Let there be unlimited capacity at the warehouses. Then, if the transportation cost function T_i are concave and $T_i(0) = 0$, there exists an optimal solution in which each retailer will receive a shipment from at most one warehouse.*

Proof. The total transportation cost is expressed as

$$\min \left\{ \sum_{i=1}^m T_i(x_i) \right\} = \min \left\{ T_1(x_1) + T_2(x_2) + \ldots + T_m(x_m) \right\} \tag{5.8}$$

Let Q denote the total amount required at the retailer. Then

$$\sum_{i=1}^m x_i = Q$$

Now let

$$x_i = a_i Q \qquad \forall i \tag{5.9}$$

where $0 \leq a_i \leq 1$, and $\sum_{i=1}^{m} a_i = 1$. Substitute Equation 5.9 in Equation 5.8,

$$\begin{aligned}
\min \left\{ \sum_{i=1}^{m} T_i(x_i) \right\} &= \min \{T_1(a_1Q) + T_2(a_2Q) + \ldots + T_m(a_mQ)\} \\
\text{(by concavity property)} &\leq \min \{a_1T_1(Q) + a_2T_2(Q) + \ldots + a_mT_m(Q)\} \\
&\leq \min \{T_1(Q), T_2(Q), \ldots, a_mT_m(Q)\}
\end{aligned}$$

Thus, the optimal solution is the least cost among the m warehouses. ■

The theorem suggests that the shipment to any retailer R_j is coming from only one warehouse, say $W_{i_j^*}$, where the minimum cost occurs. Thus, the solution to each uncapacitated single retailer problem (5.7) is to choose the minimum total expected cost option among *m+1* options (m options of ordering from only one warehouse out of m warehouses and one option from ordering nothing). If the solution is to place an order, then the warehouse i_j^* will send the shipment to retailer j.

To determine the shipping amount, we can drop the fixed transportation cost from (5.7) first. Then we use the classical newsboy problem formula to obtain the shipment size. For every warehouse i, the solution to (5.7) can be expressed as follows.

$$\Phi_D(x_i^*) = \frac{p - v_i}{h + p} \tag{5.10}$$

The cost associated with this decision is

$$(h+p)\int_0^{x_i^*} \Phi_D(\xi)d\xi + p\bar{D} - (p - \lambda_i)x_i^* + F_i + v_i x_i^* \tag{5.11}$$

If the decision is not 'not ordering", its associated cost would be $p\bar{D}$.

The procedure to solve $\min_{x \in \mathbf{X}} L(\mathbf{x}, \lambda)$ is summarized as follows.

Step 1: Find the most economical option among *m+1* options for every retailer and let it be denoted by TC_j^*.

$$TC_j^* =$$

$$\min \Big(p_j \bar{D}_j, \min_{i=1,\dots,m} \Big((h_j + p_j) \int_0^{x_{ij}^*} \Phi_{D_j}(\xi_j) d\xi_j + p_j \bar{D}_j - (p_j - \lambda_i) x_{ij}^* + v_{ij} x_{ij}^* + F_{ij} \Big)\Big)$$

Note that if the decision is to order, then we will get the index i_j^* and the cost associated with this decision in the current step.

Step 2: Sum up cost from all retailers and subtract with the term $\sum_{i=1}^{m} \lambda_i C_i$.

$$L(\mathbf{x}, \lambda) = \sum_{j=1,\dots,n} TC_j^* + \sum_{i=1}^{m} \lambda_i C_i$$

5.5 Computational Results

The objective of the computation experiment is to obtain lower bounds in order to measure the upper bound quality obtained by employing DSSP heurisctic from the previous chapter. To obtain the best lower bound, we implement the modified subgradient optimization algorithm discussed in previous section. The subgradient algorithm is programmed in C. The experiments have been performed on the same set of test problems in the previous chapter.

We set the subgradient algorithm stopping criteria parameters as follows.

$$\varepsilon = 10^{-7}$$

or

$$t_{max} = 3000$$

It means that the algorithm would terminate if the differences between Lagrangian multipliers is less than 10^{-7} for five consecutive iterations or the number of iterations exceeds *3,000*. In addition, while both stopping criteria has not been reached yet, if the lower bound (*LBD*) does not improve for 25 consecutively iterations, then we will halve the step scale (d_t).

To determine the *UBD* in *step 5* (for computing the step size) of the subgradient method, we assume that

$$x_{ij} = \Phi^{-1}(\frac{p_j - v_{ij}}{h_j + p_j})/m$$

Then the *UBD* is calculated from

$$UBD = \sum_{j=1}^{n}\{(h_j + p_j)\int_0^{\sum_{i=1}^{m} x_{ij}} \Phi_{D_j}(\xi_j)d\xi_j + p_j\bar{D}_j - p_j(\sum_{i=1}^{m} x_{ij})\} + \sum_{j=1}^{n}\sum_{i=1}^{m}(F_{ij} + v_{ij}x_{ij})$$

For every warehouse-retailer combination problem, the upper bound is compared with two lower bounds: the initial and the best lower bound. The initial lower bound which is obtained by letting all Lagrange multipliers be zero, $\lambda = 0$. Then the subgradient algorithm is applied to obtain the best lower bound. For each warehouse-retailer combination problem, there are five upper bounds to be compared. Each upper bound is an estimation to the optimal value with with different number of scenarios in the approximated problem.

The computational results are reported in Table 5.1 to ??. The first three columns represent the number of *warehouses* (W), *retailers* (R), and *scenarios* (S) respectively. The following two columns report the average, minimum, maximum value of the percentage of the relative gap $\%GAP$ comparing the $DSSP$ heuristic solution, f_{DSSP}, with the best (maximum) lower bound from the Lagrangian relaxation model f_{LR} which is obtained from the subgradient method. The relative gap percentage is calculated from

$$\%GAP = (\frac{f_{DSSP} - f_{LR}}{f_{LR}}) * 100\%.$$

The results show that for most of the problems in both demand distributions, the average $\%GAP$ behaves inversely with the number of scenarios in the approximated problem. That means if the number of scenarios increases, then the average $\%GAP$ decreases. In other words, DSSP can provide a better estimation to the optimal value when the distribution demand is approximated with more number of scenarios. The results also indicate that the average $\%GAP$ goes with the number of warehouses. Furthermore, it appears in many problems that the $\%GAP$ of the exponential demand distribution is smaller than the uniform demand distribution. It implies that the the DSSP estimates the better upper bound for the exponential distribution. The overall average $\%GAP$ reported for all test problems are 3.75% and 2.67% for the uniform and exponential demand distribution respectively. The largest gap reported is 15.47% for the uniformly distributed demand and 9.09% for the exponentially distributed demand. The smallest gap reported is 0.32% for the uniformly distributed demand and 0.46% for the exponentially distributed demand.

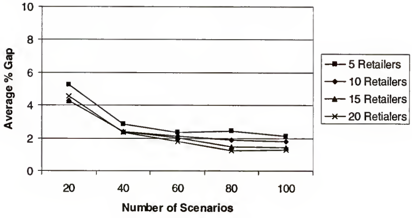
Figure 5.1- 5.4 are the line graphs showing the relationship between the average $\%GAP$ and the number of scenarios. Each figure represents the problem of a particular number of warehouses with various number of retailers in the problem except for the large problem graph. In the two, four, and six warehouses problems, each line in the graph represents the number of retailers in the problem. In the large problem graph, each line represents a particular problem size. There are two graphs in each figure: one for the uniformly, and another one for the exponentially distributed demand. All figures show that there is less variability in the average $\%GAP$ among the number of retailers for the exponential demand. It is easily to see that the line graphs stay close together in the exponential demand graphs compared to the uniform demand graphs. That means the number of retailers has more effect in the uniform demand case than the exponentially one.

Table 5.1: Summary of % GAP for two warehouses with various number of retailers problems with uniformly distributed demands

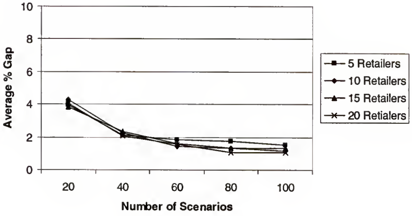
Size		Scenarios	%GAP		
W	R		Average	Min.	Max.
2	5	20	**5.2705**	2.1947	8.3672
		40	**2.8661**	0.5144	5.3046
		60	**2.3473**	0.3237	7.2278
		80	**2.4486**	0.4749	3.7548
		100	**2.1089**	0.6396	5.2497
2	10	20	**4.2828**	2.4421	9.0450
		40	**2.4131**	1.0785	5.3973
		60	**2.1173**	0.4425	3.9709
		80	**1.9033**	1.2114	2.9769
		100	**1.7756**	1.0815	3.6845
2	15	20	**4.2896**	2.1811	6.0484
		40	**2.3952**	0.9198	3.5147
		60	**2.0099**	1.2727	3.1396
		80	**1.4892**	0.8115	2.7431
		100	**1.4213**	0.7224	1.8799
2	20	20	**4.5427**	2.4838	6.8975
		40	**2.3394**	1.6414	3.8105
		60	**1.7766**	1.1182	3.1927
		80	**1.2272**	0.5507	1.9643
		100	**1.3011**	0.7467	1.9137

Table 5.2: Summary of % GAP for two warehouses with various number of retailers problems with exponentially distributed demands

Size		Scenarios	%GAP		
W	R		Average	Min.	Max.
2	5	20	**3.9758**	2.0715	6.5749
		40	**2.1036**	0.4584	3.7732
		60	**1.8275**	0.5179	3.4639
		80	**1.7294**	1.0000	3.4693
		100	**1.5203**	0.6977	2.9473
2	10	20	**4.2632**	2.7749	7.1038
		40	**2.2597**	1.1128	3.6709
		60	**1.4194**	0.4608	2.5744
		80	**1.3361**	0.7674	1.9985
		100	**1.1641**	0.5850	1.7388
2	15	20	**3.8097**	1.4686	5.7992
		40	**2.3610**	1.2213	3.2544
		60	**1.5932**	1.0595	2.1216
		80	**1.3221**	0.7468	2.0919
		100	**1.3383**	0.6574	1.9184
2	20	20	**4.0781**	1.5664	6.8344
		40	**2.0801**	1.4231	3.1403
		60	**1.5680**	1.0088	1.9857
		80	**1.0642**	0.5010	1.8217
		100	**1.0405**	0.7005	1.3780



(a) Uniform



(b) Exponential

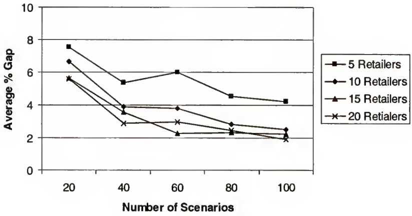
Figure 5.1: Average %Gap (best LBD) for two warehouses

Table 5.3: Summary of % GAP for four warehouses with various number of retailers problems with uniformly distributed demands

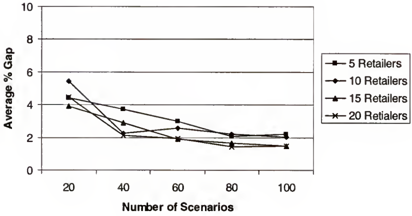
Size		Scenarios	%GAP		
W	R		**Average**	Min.	Max.
4	5	20	**7.5432**	4.2240	15.4659
		40	**5.3799**	2.6050	7.8394
		60	**6.0131**	2.6515	8.9747
		80	**4.5558**	2.6667	8.5065
		100	**4.2203**	2.3575	7.2226
4	10	20	**6.6453**	2.3388	9.5158
		40	**3.8964**	1.7273	6.6988
		60	**3.8021**	1.3788	7.1381
		80	**2.8345**	1.4248	4.1387
		100	**2.5063**	1.8717	3.0253
4	15	20	**5.6505**	2.8876	9.2673
		40	**3.5537**	2.3161	4.9729
		60	**2.2577**	1.2677	3.3329
		80	**2.3171**	1.3290	3.0473
		100	**2.2411**	1.3115	4.2550
4	20	20	**5.6052**	2.9856	8.8079
		40	**2.8911**	1.8466	5.5997
		60	**2.9502**	1.4567	3.7831
		80	**2.4564**	1.5558	3.6724
		100	**1.8986**	0.7881	3.6035

Table 5.4: Summary of % GAP for four warehouses with various number of retailers problems with exponentially distributed demands

Size		Scenarios	%GAP		
W	R		Average	Min.	Max.
4	5	20	**4.43307**	1.61710	6.71725
		40	**3.74925**	0.66155	6.95340
		60	**3.01109**	0.94744	6.00608
		80	**2.05161**	0.49588	4.32933
		100	**2.20688**	0.97494	3.31930
4	10	20	**5.43776**	2.81965	9.08982
		40	**2.25514**	0.84634	4.81623
		60	**2.55883**	0.80639	5.55310
		80	**2.19358**	1.38808	3.20357
		100	**2.02194**	0.98386	3.10035
4	15	20	**3.92325**	1.45855	5.92698
		40	**2.88050**	1.96739	4.30516
		60	**1.87539**	1.44528	2.53805
		80	**1.67920**	0.47450	3.34270
		100	**1.46930**	0.67668	2.29069
4	20	20	**4.45924**	2.58867	6.79717
		40	**2.11634**	1.04878	3.74961
		60	**1.95216**	0.99052	2.96083
		80	**1.44035**	0.69649	2.26821
		100	**1.47703**	0.77251	2.31056



(a) Uniform



(b) Exponential

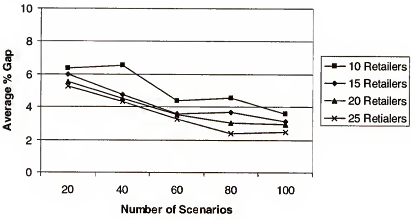
Figure 5.2: Average %Gap (best LBD) for four warehouses

Table 5.5: Summary of % GAP for six warehouses with various number of retailers problems with uniformly distributed demands

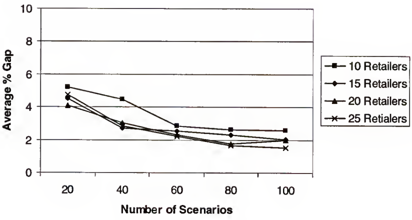
Size		Scenarios	%GAP		
W	R		**Average**	Min.	Max.
6	10	20	**6.35337**	4.79593	8.15290
		40	**6.54585**	3.92919	10.06927
		60	**4.39570**	3.02571	6.91162
		80	**4.56873**	3.03198	6.57606
		100	**3.61513**	1.73449	5.79110
6	15	20	**6.01000**	2.97233	9.65109
		40	**4.74029**	2.33175	6.94994
		60	**3.59946**	2.50648	5.28953
		80	**3.70633**	2.35134	5.04463
		100	**3.15663**	1.94358	4.38206
6	20	20	**5.50709**	2.40467	7.68615
		40	**4.49438**	3.08043	5.26264
		60	**3.54298**	2.15763	5.90731
		80	**3.03590**	1.63801	4.15084
		100	**2.93561**	2.04458	5.02530
6	25	20	**5.25893**	3.09068	8.08560
		40	**4.31816**	3.40861	5.87747
		60	**3.27204**	1.95002	4.30819
		80	**2.41209**	1.58404	3.16097
		100	**2.49047**	1.49162	3.85256

Table 5.6: Summary of % GAP for six warehouses with various number of retailers problems with exponentially distributed demands

Size		Scenarios	%GAP		
W	R		**Average**	Min.	Max.
6	10	20	**5.1844**	3.0374	7.3217
		40	**4.4792**	3.0666	6.3712
		60	**2.8532**	1.0308	4.5031
		80	**2.6185**	1.2381	5.3927
		100	**2.5839**	1.0358	3.3324
6	15	20	**4.5123**	2.9377	7.0596
		40	**2.7071**	0.9780	4.5773
		60	**2.5131**	1.5962	3.8610
		80	**2.2996**	1.7520	3.2678
		100	**2.0474**	1.4501	3.1322
6	20	20	**4.0925**	2.2568	6.5534
		40	**3.0229**	1.5473	4.3658
		60	**2.2977**	1.5256	3.5423
		80	**1.7468**	1.2222	2.3836
		100	**1.9640**	1.2248	3.1099
6	25	20	**4.7347**	2.5655	7.3597
		40	**2.8712**	2.1149	4.0509
		60	**2.2124**	1.4247	3.1610
		80	**1.6719**	0.8583	2.2778
		100	**1.5077**	1.2253	2.0017



(a) Uniform



(b) Exponential

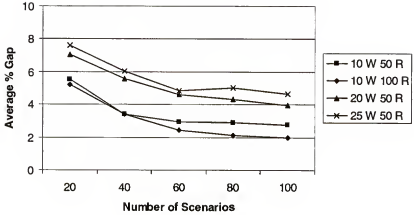
Figure 5.3: Average %Gap (best LBD) for six warehouses

Table 5.7: Summary of % GAP for large problems with uniformly distributed demands

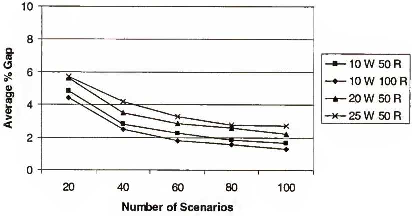
Size		Scenarios	%GAP		
W	R		**Average**	Min.	Max.
10	50	20	**5.5157**	5.0315	6.0444
		40	**3.4126**	2.3851	4.3306
		60	**2.9608**	2.0042	3.7984
		80	**2.9218**	2.0405	3.5223
		100	**2.7513**	2.4068	3.3224
10	100	20	**5.2196**	4.3332	6.2107
		40	**3.3914**	2.5652	3.9007
		60	**2.4356**	1.8034	2.9868
		80	**2.1334**	1.5379	2.7518
		100	**1.9974**	1.3347	2.5310
20	50	20	**7.0553**	5.7918	8.2267
		40	**5.5698**	4.6696	7.1511
		60	**4.5981**	3.8933	5.7933
		80	**4.3412**	3.5291	5.7252
		100	**3.9715**	3.2088	4.8972
25	50	20	**7.6140**	6.3478	8.6434
		40	**6.0406**	5.1851	6.8680
		60	**4.8569**	4.1901	5.6154
		80	**5.0438**	4.4815	5.6444
		100	**4.6679**	3.8926	5.6179

Table 5.8: Summary of % GAP for large problems with exponentially distributed demands

Size		Scenarios	%GAP		
W	R		**Average**	Min.	Max.
10	50	20	**4.8406**	4.1123	6.7421
		40	**2.8171**	2.0137	3.9764
		60	**2.2431**	1.8896	2.8806
		80	**1.8559**	1.4454	2.2157
		100	**1.6693**	1.3541	2.3221
10	100	20	**4.4455**	3.7486	5.3499
		40	**2.4968**	2.0305	3.0085
		60	**1.8160**	1.4339	2.2268
		80	**1.5654**	1.0468	1.8922
		100	**1.2940**	1.0274	1.5901
20	50	20	**5.6134**	4.0778	7.2085
		40	**3.5057**	3.0523	4.2722
		60	**2.8651**	2.1955	3.4713
		80	**2.5795**	2.0705	3.5085
		100	**2.2224**	1.5391	2.6677
25	50	20	**5.7156**	4.5628	8.5230
		40	**4.1840**	3.7080	4.7249
		60	**3.2670**	2.5124	4.1175
		80	**2.7787**	2.0165	3.8310
		100	**2.7311**	2.0514	3.4243



(a) Uniform



(b) Exponential

Figure 5.4: Average %Gap (best LBD) for large problems

5.6 Conclusion

This chapter measures the upper bound quality obtained from the previous chapter using *DSSP* heuristic. The criterion used is the gap between upper bounds (*DSSP* solution) and the best lower bounds. The lower bounds are derived from the Lagrangian relaxation model. The relaxed problem can be decomposed into n single-retailer uncapacitated subproblems. For a fixed value of Lagrange multiplier, the solution to the subproblem can be retrieved by using the classical newsboy problem formula. Then the subgradient algorithm is employed to solve the Lagrangian dual problem to obtain the best lower bound. The experiments have been performed on various problem sizes with two different demand distributions as in the previous chapter. For both distributions, the number of scenarios affects the *DSSP* solution quality. The result shows that if the number of scenarios increases, then DSSP can provide a better estimation to the optimal value ($\%GAP$ is smaller). The overall average percentage gap of the exponentially distributed demand is little better than the uniform one. Finally, the computational results have demonstrated that the upper bounds obtained from the previous chapter can be used to estimate the optimal solution to the capacitated inventory and transportation problem.

CHAPTER 6
THE LAGRANGIAN RELAXATION BASED HEURISTIC

6.1 Introduction

This chapter presents a new approach for a fixed charge transportation model without using the scenario generating technique as in Chapter 4. This new approach employs the Lagrangian relaxation (LR) model to find an optimal solution to the linear transportation cost problem. Unlike the scenario-based DSSP heuristic, the Lagrangian relaxation based DSSP heuristic uses the true demand distribution instead of an approximation of the demand distribution (using scenarios). After obtaining an optimal solution of the linear transportation cost model, the linear cost factor is updated as in the DSSP heuristic.

The LR-based heuristic obtains the solution by solving a series of linear transportation cost problems (nonlinear) instead of solving linear programming problems (approximated problems) as in the scenario based heuristic. The LR-based heuristic is better estimating an optimal solution because the DSSP heuristic is applied directly to the original problem not through the approximated problem.

The material of this chapter is organized along the following sequence. Section 6.2 reviews the linear transportation cost model and its Lagrangian relaxation model. Section 6.3 outlines the solution procedure. Section 6.4 focuses on the computational experiment. Finally, the concluding remark is presented in Section 6.5.

6.2 A Linear Transportation Cost Model

6.2.1 Model Formulation

Let us revisit the formulation of a system of m warehouses n retailers with a linear transportation cost.

$$
\begin{aligned}
\min \sum_{j=1}^{n} & \left(h_j \int_0^{\sum_{i=1}^m x_{ij}} (\sum_{i=1}^{m} x_{ij} - \xi_j)\varphi_{D_j}(\xi_j)d\xi_j \right. \\
& \left. + p_j \int_{\sum_{i=1}^m x_{ij}}^{\infty} (\xi_j - \sum_{i=1}^{m} x_{ij})\varphi_{D_j}(\xi_j)d\xi_j \right) \\
& + \sum_{j=1}^{n}\sum_{i=1}^{m}(v_{ij}x_{ij})
\end{aligned}
\tag{6.1}
$$

$$
\begin{aligned}
s.t. \quad \sum_{j=1}^{n} x_{ij} & \leq C_i \quad \forall i \\
x_{ij} & \geq 0 \quad \forall i,j
\end{aligned}
$$

The objective is to minimize the inventory cost and transportation cost subject to the capacity and nonnegativity constraints. By rearranging terms, problem (6.1) is equivalent to

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} \left\{ (h_i + p_i) \int_0^{\sum_{i=0}^m x_{ij}} \Phi_{D_j}(\xi_j)d\xi_j + p_j\bar{D}_j - \sum_{i=1}^{m} p_j x_{ij} \right\} + \sum_{j=1}^{n}\sum_{i=1}^{m} v_{ij}x_{ij} \\
s.t. \quad & \sum_{j=1}^{n} x_{ij} \leq C_i \quad \forall i \\
& x_{ij} \geq 0 \quad \forall i,j
\end{aligned}
$$

6.2.2 A Lagrangian Relaxation Model

Let λ_i, for $i = 1, \ldots, m$, be a Lagrange multiplier corresponding to the i^{th} capacity constraint. The Lagrangian function is expressed as follows.

$$F(x_{ij}, \lambda_i) = \sum_{j=1}^{n} \left\{ (h_j + p_j) \int_0^{\sum_{i=0}^{m} x_{ij}} \Phi_{D_j}(\xi_j) d\xi_j + p_j \bar{D}_j - \sum_{i=1}^{m} (p_j - v_{ij} - \lambda_i) x_{ij} \right\} - \sum_{i=1}^{m} \lambda_i C_i \tag{6.2}$$

We can solve the dual problem of Lagrangian problem (6.2) to yield the optimal Lagrange multipliers, λ_i^* for $i = 1, \ldots, m$. The optimal Lagrange multipliers are distinguished into two sets by their optimal values: $\lambda_i^* > 0$ and $\lambda_i^* = 0$. Let I_{LR} denote the set of warehouses which their capacity constraints are binding, and $\tilde{I}_{LR}$ otherwise. Thus $I_{LR} = \{i : \lambda_i^* > 0\}$ and $\tilde{I}_{LR} = \{i : \lambda_i^* = 0\}$. We know that if $\lambda_i^* = 0$, the i^{th} constraint is redundant. In other words, W_i is not used at full capacity. On the other hand, if $\lambda_i^* > 0$, then the i^{th} constraint is binding. That means

$$\sum_{j=1}^{n} x_{ij}^* = C_i$$

Then the original problem becomes

$$\min \sum_{j=1}^{n} \left(h_j \int_0^{\sum_{i=1}^{m} x_{ij}} (\sum_{i=1}^{m} x_{ij} - \xi_j) \varphi_{D_j}(\xi_j) d\xi_j + p_j \int_{\sum_{i=1}^{m} x_{ij}}^{\infty} (\xi_j - \sum_{i=1}^{m} x_{ij}) \varphi_{D_j}(\xi_j) d\xi_j \right) + \sum_{j=1}^{n} \sum_{i=1}^{m} (v_{ij} x_{ij})$$

$$
\begin{aligned}
s.t. \quad \sum_{j=1}^{n} x_{ij} &= C_i \qquad \forall i \in I_{LR} \\
x_{ij} &\geq 0 \qquad \forall i, j
\end{aligned}
$$

The corresponding Lagrangian problem is

$$
\begin{aligned}
&\min \; F(x_{ij}, \lambda_i) = \\
&\quad \sum_{j=1}^{n} \left\{ (h_j + p_j) \int_0^{\sum_{i=0}^{m} x_{ij}} \Phi_{D_j}(\xi_j) d\xi_j + p_j \bar{D}_j - \sum_{i=1}^{m} p_j x_{ij} + \sum_{i=1}^{m} v_{ij} x_{ij} \right\} \\
&\quad + \sum_{i \in I_{LR}} \lambda_i \left(\sum_{j=1}^{n} x_{ij} - C_i\right) \qquad (6.3) \\
&s.t. \quad x_{ij} \geq 0
\end{aligned}
$$

or

$$
\begin{aligned}
&\min \; F(x_{ij}, \lambda_i) = \\
&\quad \sum_{j=1}^{n} \left\{ (h_j + p_j) \int_0^{\sum_{i=0}^{m} x_{ij}} \Phi_{D_j}(\xi_j) d\xi_j + p_j \bar{D}_j - \sum_{i=1}^{m} p_j x_{ij} \right. \\
&\quad \left. + \sum_{i \in I} (v_{ij} + \lambda_i) x_{ij} + \sum_{i \in \bar{I}_{LR}} v_{ij} x_{ij} \right\} - \sum_{i \in I_{LR}} \lambda_i C_i \qquad (6.4) \\
&s.t. \quad x_{ij} \geq 0
\end{aligned}
$$

Now suppose that the nonnegativity constraint is ignored. Taking the derivatives of Equation (6.4) with respect to x_{ij}, we obtain

$$
\frac{\partial F(x_{ij}, \lambda_i)}{\partial x_{ij}} = (h_j + p_j) \Phi_{D_j}\left(\sum_{i=0}^{m} x_{ij}\right) - p_j + v_{ij} + \lambda_i 1_{\{i \in I_{LR}\}} \quad \forall i, j
$$

If the nonnegative constraint is not binding, i.e. $x_{ij}^* > 0$, then

$$
(h_j + p_j) \Phi_{D_j}\left(\sum_{i=0}^{m} x_{ij}\right) - p_j + v_{ij} + \lambda_i 1_{\{i \in I_{LR}\}} = 0
$$

$$\Phi_{D_j}(\sum_{i=0}^{m} x_{ij}) = \frac{p_j - v_{ij} - \lambda_i 1_{\{i \in I_{LR}\}}}{h_j + p_j}$$

In set $\tilde{I}_{LR}$, the warehouses become uncapacitated. That means there will be at most one warehouse from this set for which $x^*_{ij} > 0$ for any j. In other words, if R_j needs to be supplied by warehouses in set $\tilde{I}_{LR}$, only one warehouse with the minimum unit transportation cost will be the supplier of R_j.

6.3 Solution Approach

This section outlines the procedure of the Lagrangian based heuristic. This procedure approximates a solution for the fixed charge transportation model by solving successive a linear transportation cost problems using the Lagrangian relaxation method. Then linear cost factor is recursively updated. The procedure composes of three main stages. The first stage finds the optimal Lagrange multipliers and its objective value using the subgradient optimization of the linear transportation cost problem. The obtained multiplier optimal values reveal which capacity constraints are binding (or which warehouse is used at full capacity). However, they do not tell us the primal optimal value, x^*_{ij}. In order to recover x^*_{ij} from the Lagrange dual problem solution, we employ the perturbation idea to investigate which warehouses are supplying a particular retailer in the second stage. Finally, the linear factor is updated based on the optimal solution obtained from the second stage.

After all multiplier optimal values are known, we can compute the following ratios.

$$RA_{ij} = \frac{p_j - v_{ij} - \lambda_i 1_{\{i \in I\}}}{h_j + p_j} \quad \text{for } \forall i, j$$

From these ratios, two cases can happen. In the first case, for any R_j, all corresponding RA_{ij}'s are different. These different ratios yield different shipment sizes. Among them, there will be only one shipment size which gives the minimum total cost. That means there is only one warehouse supplying this particular R_j. In the second case, there occur duplicated ratios. The duplicated ratios yield the same shipment sizes. It implies that the R_j will potentially be supplied by all corresponding W_i's where their ratios are equal.

In the perturbation stage, we aim to find what warehouse is supplying a particular R_j. Let $\tilde{x}_{ij}$ be a new decision variable, and $x_{ij} = \tilde{x}_{ij} + \epsilon$, where ϵ is a constant number. Assume that R_j receives the shipment size of ϵ from $W_{\acute{i}}$ in which can be any warehouse. If we want to check whether or not W_i is supplying a particular R_j, we would perturb the Lagrangian problem by appending a constraint, $x_{ij} \geq \epsilon$. If x_{ij} is indeed an optimal flow ($x_{ij} > 0$), then the Lagrangian objective value does not change. On the other hand, the objective value increases, if x_{ij} is not in the optimal solution ($x_{ij} = 0$). This causes by initiating an unnecessary ϵ flow in the solution. The perturbation of all possible flows is performed one flow at a time. Let $\acute{i}$ be a warehouse where $x_{\acute{i}j} = \epsilon$. The total cost function of each perturbed problem for any R_j is as follows:

$$(h_j + p_j) \int_0^{\tilde{x}_{ij} + x_{\acute{i}j}} \Phi_{D_j}(\xi_j) d\xi_j + p_j \bar{D}_j - p_j(\tilde{x}_{ij} + x_{\acute{i}j}) + (v_{ij} + \lambda_i)\tilde{x}_{ij} + (v_{\acute{i}j} + \lambda_{\acute{i}})x_{\acute{i}j}$$

or

$$(h_j + p_j) \int_0^{\tilde{x}_{ij} + \epsilon} \Phi_{D_j}(\xi_j) d\xi_j + p_j \bar{D}_j - p_j(\tilde{x}_{ij} + x_{\acute{i}j}) + (v_{ij} + \lambda_i)\tilde{x}_{ij} + (v_{\acute{i}j} + \lambda_{\acute{i}})\epsilon$$

Then

$$\Phi_{D_j}(\tilde{x}_{ij} + \epsilon) = \frac{p_j - v_{ij} - \lambda_i}{h_j + p_j}$$

$$\tilde{x}_{ij} = \Phi_{D_j}^{-1}\left(\frac{p_j - v_{ij} - \lambda_i}{h_j + p_j}\right) - \epsilon$$

It should be noted that if $p_j - v_{ij} - \lambda_i < 0$, then $\tilde{x}_{ij} = 0$ and the total cost becomes

$$p_j\bar{D}_j + (v_{ij} + \lambda_i)\epsilon$$

For the case that all RA_{ij}'s are different, if the perturbed problem objective value is equal to the subgradient objective value in the first stage, it means that $x^*_{ij} > 0$ and x^*_{ij} satisfies

$$\Phi_{D_j}(x^*_{ij}) = \frac{p_j - v_{ij} - \lambda_i 1_{\{i \in I_{LR}\}}}{h_j + p_j}$$

As mentioned earlier, retailers with duplicated ratios will potentially have multiple shipments from warehouses with the same ratios. At most one of these warehouses is not used at full capacity and the rest is in the set I_{LR}. In addition, some of these warehouses may supply other retailers which have split shipment. That means it is not necessary that there is only one split shipment retailer for all warehouses in set I_{LR}. However, at least one of warehouses in this set has only one split shipment retailer. Since all other shipments (flows) are known, the remaining one with the split shipment retailer is calculated through the capacity constraint. To demonstrate, let $x^*_{i\hat{j}}$, $i \in I_{LR}$ be the split shipment at any W_i. The remaining flow at that warehouse is calculated as follows:

$$x^*_{i\hat{j}} = C_i - \sum_{j \in J \setminus \{\hat{j}\}} x^*_{ij} \quad \forall i \in I_{LR}$$

Let $x^*_{\hat{i}j}$, $\hat{i} \in \tilde{I}_{LR}$ be the remaining one flow from the noncapacited warehouse of R_j. This flow is given as follows:

$$\Phi_{D_j}\Big(\sum_{i \in I_{LR}} x^*_{ij} + x^*_{\hat{i}j}\Big) = \frac{p_j - v_{\hat{i}j}}{h_j + p_j} \left(= \frac{p_j - v_{ij} - \lambda_i 1_{\{i \in I_{LR}\}}}{h_j + p_j} \text{ for } \forall i \in I_{LR} \text{ with } x^*_{ij} > 0 \right)$$

The Lagrangian based heuristic is summarized as follows:

Step 0 Initialize the "linear factor", $\bar{v}^0_{ij}$.

$$\bar{v}^0_{ij} = v_{ij}$$

Step 1 Solve a linear transportation cost using the Lagrangian relaxation method. In this step, the optimal Lagrange multipliers and the objective value are determined using the subgradient optimization technique.

Step 2 Compute the ratios, RA_{ij} for $\forall i, j$ to divide retailers into two sets: different-ratio retailer and duplicated-ratio retailer.

Step 3 Perturb the Lagrangian problem with a new constraint

$$x_{ij} \geq \epsilon$$

Then compare the perturbed problem objective value with the subgradient objective value obtained from Step 1. If they are equal, then the perturbing flow is greater than zero (i.e., W_i is in fact supplying R_j). Thus, $\tilde{x}_{ij} > 0$ and $x_{ij} = \tilde{x}_{ij} + \epsilon$. Otherwise, $\tilde{x}_{ij} = 0$.

Repeat Step 3 for all possible flows (x_{ij}), or until the supplying warehouse for all retailers is found.

Step 4 Recover the primal solution, x^*_{ij}, calculate the (fixed charge) objective value, and keep the best value found.

Step 5 Update the "linear factor", $\bar{v}^{k+1}_{ij}$, at iteration *k+1*

$$\bar{v}^{k+1}_{ij} = \begin{cases} v_{ij} + \frac{F_{ij}}{\bar{x}^k_{ij}}, & \text{if } \bar{x}^k_{ij} > 0 \quad \text{at } k > 0 \\ v_{ij}, & \text{if } \bar{x}^k_{ij} = 0 \quad \text{at } k = 0 \\ \bar{v}^r_{ij}, & \text{if } \bar{x}^k_{ij} = 0 \quad \text{at } k \geq 1 \end{cases}$$

where r is the index of the most recent value of the slope scaling factor when $\bar{x}_{ij}r > 0$.

Step 6 Observe the solution, $\bar{x}^{k+1}_{ij}$. If $\bar{x}^{k+1}_{ij} = \bar{x}^k_{ij}$ (indicate no further improvement) or if the (fixed charge) objective value does not improve for five consecutive iterations, then terminates. Otherwise go to step 1.

6.4 Computational Experiments

This section compares the effectiveness of the Lagrangian based heuristic with the scenario based heuristic presented in Chapter 4. Two key factors in comparison are $\%GAP$, which is the distance of the heuristic solution from the best lower bound, and the computational times. The Lagrangian heuristic is coded in C.

The proposed heuristic is tested on three sets of warehouse-retailer combinations: two warehouses, four warehouses, and six warehouses with various number of retailers. For every warehouse-retailer combination, 50 problem instances with uniform demand distribution are tested. These test problems are generated based on the parameters described in Chapter 4. The number of scenarios in the test problems is 100.

The computational result is reported in Table 6.1. The first two columns are the number of *warehouses* (W) and *retailers* (R) respectively. The following six columns report the average, minimum, and maximum value of the percentage differences from the best lower bound for the Lagrangian based heuristic and scenario based heuristic respectively. Let f_* be an objective value attained from the heuristic. Let BLB denote the best lower bound obtaining from the Lagrangian relaxation method described in Chapter 5. The percentage differences are calculated as follows:

$$\%GAP = \left(\frac{f_* - BLB}{BLB}\right) * 100\%$$

The last six columns report the average, minimum, and maximum CPU times used by each heuristic.

The results show that the Lagrangian based heuristic found almost optimal and optimal solution. It provides better solutions than the scenario based heuristic in less CPU times (see Figure 6.1 and 6.2). As the number of retailers increases, the CPU time differences increases as well. Especially in the two warehouse problem, the Lagrangian based heuristic performed much superior than the scenario one. While the worst $\%GAP$ reported is 6.37% by using the Lagrangian based heuristic, it is 11.75% by using the scenario based heuristic.

Table 6.1: Summary of computational results

		%*GAP*						CPU Time (seconds)					
Size		LR-based Heuristic			Scenario-based Heuristic			LR-based Heuristic			Scenario-based Heuristic		
W	R	**Average**	Min	Max	**Average**	Min	Max	**Average**	Min	Max	**Average**	Min	Max
2	5	**0.6918**	0.0000	2.3206	**2.1168**	0.1076	6.5927	**0.051**	0.010	0.140	**0.158**	0.120	0.200
	10	**0.4558**	0.0000	1.4187	**1.3720**	0.4190	3.4401	**0.107**	0.020	0.270	**0.466**	0.320	0.570
	15	**0.3643**	0.0812	0.8328	**1.4327**	0.5406	3.2713	**0.195**	0.030	0.510	**0.799**	0.540	1.000
	20	**0.3377**	0.0000	0.9266	**1.3274**	0.5500	3.0129	**0.281**	0.040	0.520	**1.177**	0.720	1.470
4	5	**2.9487**	0.3692	6.3745	**4.5651**	0.9147	11.7458	**0.238**	0.070	0.520	**0.209**	0.160	0.250
	10	**1.9763**	0.7028	3.1749	**2.9756**	1.1154	6.5190	**0.505**	0.250	1.340	**0.564**	0.450	0.670
	15	**1.4034**	0.3751	3.0283	**2.3246**	0.8390	3.9530	**0.692**	0.370	1.370	**0.992**	0.680	1.340
	20	**1.0752**	0.1625	1.8594	**1.9753**	0.8016	3.2511	**0.912**	0.520	2.070	**1.501**	1.170	1.800
6	10	**3.3739**	0.8858	5.9453	**4.6565**	2.3361	8.3894	**0.804**	0.500	1.760	**0.771**	0.580	0.970
	15	**2.8960**	1.2670	4.8970	**3.6317**	1.8144	6.4853	**1.033**	0.760	2.340	**1.530**	0.990	2.010
	20	**2.0696**	0.5912	3.1941	**2.9631**	1.3472	4.7412	**1.718**	1.060	4.290	**2.089**	1.450	2.670
	25	**1.8841**	1.0214	3.4263	**2.6436**	1.2485	4.3431	**1.891**	1.340	3.950	**3.846**	2.760	5.170

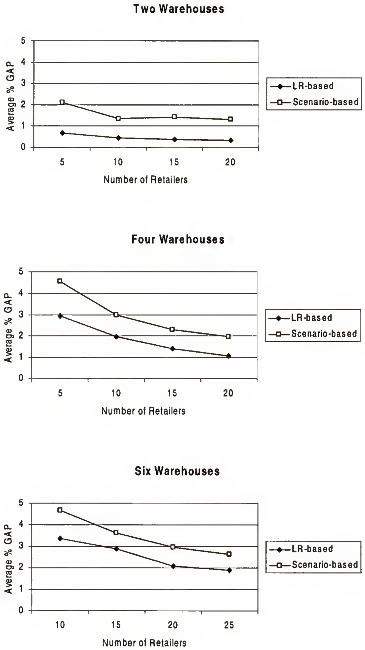


Figure 6.1: Comparison of %GAP from the best lower bound

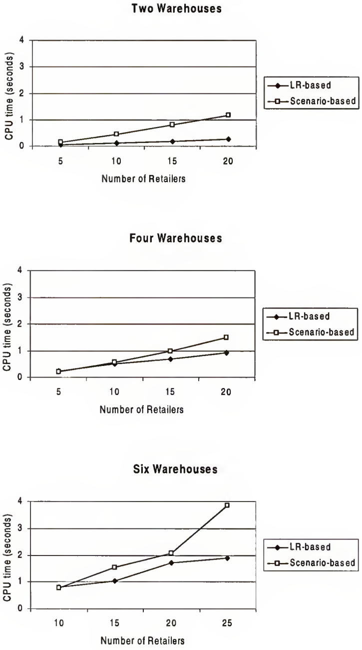


Figure 6.2: Comparison of CPU times

6.5 Concluding Remarks

This chapter develops a new Lagrangian based DSSP heuristic. This new heuristic eliminates the scenario generating scheme. It approximates the optimal solution from the true demand distribution not from an approximation of demand distribution as in the scenario based DSSP heuristic. Based on a set of test problems, the result shows that the new heuristic generates better solutions in less CPU time.

CHAPTER 7
FUTURE RESEARCH

This dissertation investigates the issues of optimizing an integrated inventory and transportation decision for a multi-warehouse multi-retailer distribution system in the face of uncertainties. The primary contributions of this dissertation are

- The development of an optimization model that extends the single-period stochastic inventory problem to accommodate a transportation cost and capacity restrictions.
- The investigation of the heuristic performance through the computational experiments.

The current optimization model is developed for a single-commodity single-period problem concerning only a distribution system in the supply chain. This allows for several model extensions. Three main directions for future research are

1. Generalize transportation cost structures
2. Extend to handle multiple commodities and multiple time periods
3. Incorporate the production cost

In this dissertation, only linear and fixed charge cost structures are examined. The more complex cost structure such as the piecewise linear concave has not been addressed. This cost structure is often found in practice when all units quantity discount structures are applied. The piecewise linear concave cost structures contains breakpoints at which the slope of the variable cost per unit changes, denoted by bk_{ij}^{k}'s in Figure 7.1. The fixed

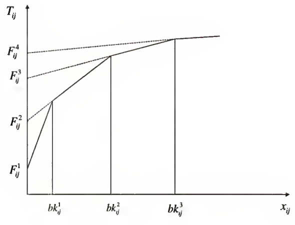


Figure 7.1: Piecewise linear concave cost structure

charge cost structure is the piecewise linear concave cost with only one slope interval. Thus the cost in each interval can be represented by a fixed charge cost function. In addition, an important property of the piecewise linear concave cost is that the unit cost slope decreases as the interval contains a larger shipment. For example, the slope in interval $(0,\, bk_{ij}^1)$ is greater than the slope in the interval $(bk_{ij}^1,\, bk_{ij}^2)$ and so on.

As mentioned before, the optimization model in this dissertation concerns only single-commodity single-period problem. This model should extend to handle multiple commodities and multiple time periods. Still, various issues within this integrated inventory-transportation model are worth exploring. These include positive leadtimes, backlogging, and so on.

Finally, the management of the supply chain operations is not complete without the production activity. The future research should integrate the production planning, inventory, and transportation decisions.

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BIOGRAPHICAL SKETCH

Paveena Chaovalitwongse was born in 1971 in Thailand. She was awarded a bachelor's degree in industrial engineering from Chulalongkorn University in Bangkok. After graduation, she joined the department as a faculty member. In 1995, she was admitted to the master's degree program at the University of Florida. After her master's studies she pursued her Ph.D. also at the University of Florida, and received her doctoral degree in December 2000.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Boghos D. Sivazlian, Chairman
Professor of Industrial and Systems Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Panos M. Pardalos , Cochair
Professor of Industrial and Systems Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Mark C. Yang
Professor of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Sherman X. Bai
Associate Professor of Industrial and Systems Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

H. Edwin Romeijn
Assistant Professor of Industrial and Systems Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Diane A. Schaub
Assistant Professor of Industrial and
Systems Engineering

This dissertation was submitted to the Graduate Faculty of the College of Engineering and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December 2000

M. J. Ohanian
Dean, College of Engineering

Winfred M. Phillips
Dean, Graduate School